Local Computing By Partial Quantifier Elimination

Eugene Goldberg

eu.goldberg@gmail.com

Abstract. Localization of computations plays a crucial role in solving hard problems efficiently. We will refer to the techniques implementing such localization as *local computing*. We relate local computing with *partial quantifier elimination* (PQE). The latter is a generalization of regular quantifier elimination where one can take a *part* of the formula out of the scope of quantifiers. The objective of this paper is to show that PQE can be viewed as a language of local computing and hence building efficient PQE solvers is of great importance. We describe application of local computing by PQE to three different problems of hardware verification: property generation, equivalence checking and model checking. Besides, we discuss using local computing by PQE for SAT solving. Finally, we relate PQE and interpolation, a form of local computing.

1 Introduction

The complexity of many practical problems quickly grows with the problem size. So some form of *local computing* is required to reduce problem complexity. In the context of hardware verification, one can single out two types of local computing. The first type is *functionally-local* computing where only a part of the *search space* is involved. An example of this type of local computing is testing. The second type is *structurally-local* computing where the algorithm operates only on the parts of the formula at hand that matter. An example of the second type of local computing is conflict analysis in SAT solving [1]. Arguably, almost all efficient algorithms of hardware verification employ local computing to partial quantifier elimination (PQE) to show that building efficient PQE solvers is of great importance.

PQE is a generalization of regular quantifier elimination (QE) that is defined as follows [2]. Let F(X, Y) be a propositional formula in conjunctive normal form¹ (CNF) where X, Y are sets of variables. Let G be a subset of clauses of F. Given a formula $\exists X[F]$, the PQE problem is to find a quantifier-free formula H(Y) such that $\exists X[F] \equiv H \land \exists X[F \setminus G]$. In contrast to full QE, only the clauses of G are taken out of the scope of quantifiers hence the name partial QE. In this

¹ Given a CNF formula F represented as the conjunction of clauses $C_0 \wedge \cdots \wedge C_k$, we will also consider F as the *set* of clauses $\{C_0, \ldots, C_k\}$.

paper, we consider PQE for formulas with only *existential* quantifiers. We will refer to H as a solution to PQE. Note that QE is just a special case of PQE where G = F and the entire formula is unquantified. (As we show in Section 8, interpolation can also be viewed as a special case of PQE.) The appeal of PQE is twofold. First, it can be *much more efficient* than QE if G is a small subset of F. Second, many old and new problems can be solved in terms of PQE.

The contributions of this paper are as follows. First, we relate local computing to PQE. Second, we put some earlier results on property generation and equivalence checking by PQE [3,4] in the context of local computing. So, we demonstrate that PQE can improve the *existing* methods of local computing. Third, we show that PQE can create *new* methods of local computing. Namely, we illustrate local computing by PQE by the examples of model checking and SAT solving. Fourth, we show the relation between PQE and interpolation that is a form of local computing. Although we do not run any experiments in this paper, we do mention some experiments of [3,4]. They show that even the current PQE solvers whose performance can be drastically improved can be practical.

The main body of this paper is structured as follows. (Some additional information can be found in the appendix.) In Section 2, we give basic definitions. A high-level view of PQE solving and some examples are presented in Section 3. Sections 4 and 5 describe property generation and equivalence checking by PQE in the context of local computing. In Sections 6 and 7, model checking and SAT-solving by PQE are introduced as new examples of local computing. Section 8 relates PQE and interpolation. In Section 9, we make conclusions.

2 Basic Definitions

In this section, when we say "formula" without mentioning quantifiers, we mean "a quantifier-free formula".

Definition 1. We assume that formulas have only Boolean variables. A literal of a variable v is either v or its negation. A clause is a disjunction of literals. A formula F is in conjunctive normal form (CNF) if $F = C_0 \land \cdots \land C_k$ where C_0, \ldots, C_k are clauses. We will also view F as the set of clauses $\{C_0, \ldots, C_k\}$. We assume that every formula is in CNF unless otherwise stated.

Definition 2. Let F be a formula. Then Vars(F) denotes the set of variables of F and $Vars(\exists X[F])$ denotes $Vars(F) \setminus X$.

Definition 3. Let V be a set of variables. An assignment \vec{q} to V is a mapping $V' \to \{0,1\}$ where $V' \subseteq V$. We will denote the set of variables assigned in \vec{q} as $Vars(\vec{q})$. We will refer to \vec{q} as a full assignment to V if $Vars(\vec{q}) = V$. We will denote as $\vec{q} \subseteq \vec{r}$ the fact that a) $Vars(\vec{q}) \subseteq Vars(\vec{r})$ and b) every variable of $Vars(\vec{q})$ has the same value in \vec{q} and \vec{r} .

Definition 4. A literal and a clause are said to be **satisfied** (respectively **falsified**) by an assignment \vec{q} if they evaluate to 1 (respectively 0) under \vec{q} .

Definition 5. Let C be a clause. Let H be a formula that may have quantifiers, and \vec{q} be an assignment to Vars(H). If C is satisfied by \vec{q} , then $C_{\vec{q}} \equiv 1$. Otherwise, $C_{\vec{q}}$ is the clause obtained from C by removing all literals falsified by \vec{q} . Denote by $H_{\vec{q}}$ the formula obtained from H by removing the clauses satisfied by \vec{q} and replacing every clause C unsatisfied by \vec{q} with $C_{\vec{q}}$.

Definition 6. Let G, H be formulas that may have existential quantifiers. We say that G, H are **equivalent**, written $\mathbf{G} \equiv \mathbf{H}$, if $G_{\vec{q}} = H_{\vec{q}}$ for all full assignments \vec{q} to $Vars(G) \cup Vars(H)$.

Definition 7. Let F(X, Y) be a formula and $G \subseteq F$ and $G \neq \emptyset$. The clauses of G are said to be **redundant** in $\exists X[F]$ if $\exists X[F] \equiv \exists X[F \setminus G]$. If $F \setminus G$ implies G, the clauses of G are redundant in $\exists X[F]$ but the opposite is not true.

Definition 8. Given a formula $\exists X[F(X,Y))]$ and G where $G \subseteq F$, the **Partial Quantifier Elimination** (**PQE**) problem is to find H(Y) such that $\exists X[F] \equiv H \land \exists X[F \setminus G]$. (So, PQE takes G out of the scope of quantifiers.) The formula H is called a solution to PQE. The case of PQE where G = F is called **Quantifier Elimination** (**QE**).

Example 1. Consider formula $F = C_0 \land \cdots \land C_4$ where $C_0 = \overline{x}_2 \lor x_3$, $C_1 = y_0 \lor x_2$, $C_2 = y_0 \lor \overline{x}_3$, $C_3 = y_1 \lor x_3$, $C_4 = y_1 \lor \overline{x}_3$. Let $Y = \{y_0, y_1\}$ and $X = \{x_2, x_3\}$. Consider the PQE problem of taking C_0 out of $\exists X[F]$ i.e., finding H(Y) such that $\exists X[F] \equiv H \land \exists X[F \setminus \{C_0\}]$. One can show that $\exists X[F] \equiv y_0 \land \exists X[F \setminus \{C_0\}]$ (see Subsection 3.3) i.e., $H = y_0$ is a solution to this PQE problem.

Definition 9. Given a formula $\exists X[F(X,Y))]$ and G where $G \subseteq F$, the decision version of PQE is to check if G is redundant in $\exists X[F]$ i.e., if $\exists X[F] \equiv \exists X[F \setminus G]$.

Definition 10. Let clauses C', C'' have opposite literals of exactly one variable $w \in Vars(C') \cap Vars(C'')$. Then C', C'' are called **resolvable** on w.

Definition 11. Let C be a clause of a formula F and $w \in Vars(C)$. The clause C is said to be **blocked** [5] in F with respect to the variable w if no clause of F is resolvable with C on w.

Proposition 1. Let a clause C be blocked in a formula F(X,Y) with respect to a variable $x \in X$. Then C is redundant in $\exists X[F]$, i.e., $\exists X[F \setminus \{C\}] \equiv \exists X[F]$.

The proofs of propositions are given in Appendix B.

3 PQE solving

In this section, we briefly describe a PQE-solver named *START* [6]. Our objective here is just to give an idea of how the PQE problem is solved. So, in Subsection 3.2, we give a high-level description omitting some details. In Subsections 3.3 and 3.4, we provide examples illustrating PQE solving.

3.1 Some background

Information on QE in propositional logic can be found in [7,8,9,10,11,12]. QE by redundancy based reasoning is presented in [13,14]. One of the merits of such reasoning is that it allows to introduce *partial* QE. A description of PQE algorithms can be found in [2,4,6]. The sources of PQE algorithms can be be downloaded from [15,16]. The PQE solver *START* described below can be viewed as an implementation of a generic PQE algorithm introduced in [4].

3.2 High-level view

START is meant for taking out a single clause. Namely, given a formula $\exists X[F(X,Y)]$ and a clause $C \in F$, START finds a formula H(Y) such that $\exists X[F] \equiv H \land \exists X[F \setminus \{C\}]$. To take a multi-clause formula G out of $\exists X[F]$, one needs to apply START |G| times taking out the clauses of G one by one. Like all existing PQE algorithms, START uses **redundancy based reasoning** justified by the proposition below. This proposition shows that to solve the PQE problem of taking a clause C out of $\exists X[F(X,Y)]$, it suffices to find a formula H(Y) implied by F that makes C redundant in $H \land \exists X[F]$. We refer to the clause that START currently tries to prove redundant as the **target** clause.

Proposition 2. Formula H(Y) is a solution to the PQE problem of taking a clause C out of $\exists X[F(X,Y)]$ (i.e., $\exists X[F] \equiv H \land \exists X[F \setminus \{C\}]$) iff

1. $F \Rightarrow H$ and 2. $H \land \exists X[F] \equiv H \land \exists X[F \setminus \{C\}]$

START finds a solution H by branching on variables of F. The idea here is to reach a subspace where C can be easily proved or made redundant in $\exists X[F]$. To express the redundancy of C in subspace \vec{q} , START generates a clause Kcalled a **certificate** that implies C in subspace \vec{q} . The certificate K has the property $\exists X[K \wedge F] \equiv \exists X[F]$. So, it can always be added to F. If the target C was used to generate K, then START adds K to F to make C redundant in subspace \vec{q} . (Because C may not be redundant in subspace \vec{q} without adding K, see the example of Subsection 3.3.) If K was derived without using C, the latter is already redundant in $\exists X[F]$ in subspace \vec{q} . So, adding K is optional. If a certificate K is added to F and it depends only on variables of Y, it is also added to the solution H. Originally, $H = \emptyset$.

If the current target clause C becomes unit² in subspace \vec{q} , START temporarily picks a different sequence of targets. Namely, START tries to prove

² An unsatisfied clause is called *unit* if it has only one unassigned literal. Due to special decision making of START (variables of Y are assigned before those of X), if the target clause C becomes unit, its unassigned variable is always in X. We assume here that C contains at least one variable of X. (Taking out a clause depending only on unquantified variables, i.e. those of Y, is trivial.)

redundancy of the clauses resolvable with C on its only unassigned variable (denote this variable as x_i). If these clauses are proved redundant in subspace \vec{q} , clause C is blocked at x_i and so is redundant in subspace \vec{q} . Otherwise, a clause implied by F and falsified by \vec{q} is derived and added to F to make C redundant in subspace \vec{q} . The fact that START changes targets means that it may need to prove redundancy of clauses other than C. The difference is that the main target (i.e., C) must be proved globally whereas the secondary targets need to be proved redundant only locally (in some subspaces). By resolving certificate clauses derived in different branches START eventually produces a certificate that simply implies C (in the entire search space). This certificate is a proof that C is redundant in the current formula $\exists X[F]$ globally. At this point H is a solution to the PQE problem.

3.3 An example of PQE solving

Here we show how START solves Example 1 introduced in Section 2. Recall that one takes C_0 out of $\exists X[F(X,Y)]$ where $F = C_0 \land \cdots \land C_4$ and $C_0 = \overline{x}_2 \lor x_3$, $C_1 = y_0 \lor x_2, C_2 = y_0 \lor \overline{x}_3, C_3 = y_1 \lor x_3, C_4 = y_1 \lor \overline{x}_3$ and $Y = \{y_0, y_1\}$ and $X = \{x_2, x_3\}$. That is, one needs to find H(Y) such that $\exists X[F] \equiv H \land \exists X[F \setminus \{C_0\}]$.

Consider branching on y_0 . In subspace $y_0 = 0$, clauses C_1, C_2 become unit. After assigning $x_2 = 1$ to satisfy C_1 , the clause C_0 turns into unit too and a conflict occurs (to satisfy C_0 and C_2 , one has to assign the opposite values to x_3). After a standard conflict analysis [1], a conflict clause $K' = y_0$ is obtained by resolving C_1 and C_2 with C_0 . Since K' is obtained using the target C_0 itself, START adds K' to F to make C_0 redundant in subspace $y_0 = 0$. (It is not hard to check that C_0 is indeed not redundant in $\exists X[F]$ in subspace $y_0 = 0$ without adding K'.) The clause K' is a certificate of redundancy of C_0 in the current formula $\exists X[F]$ in subspace $y_0 = 0$. Since K' depends only on variables of Y, it is added to the solution H.

Now consider the subspace $y_0 = 1$. Since the clause C_1 is satisfied by $y_0 = 1$, no clause of F is resolvable with C_0 on variable x_2 in subspace $y_0 = 1$. So, C_0 is blocked at variable x_2 and hence redundant in $\exists X[F]$ in subspace $y_0 = 1$. Then START generates a certificate $K'' = \overline{y}_0 \lor \overline{x}_2$ implying C_0 in subspace $y_0 = 1$. (This certificate consists of the literal \overline{y}_0 specifying the subspace where C_0 is blocked and the literal \overline{x}_2 of C_0 i.e., the literal of the variable x_2 at which C_0 is blocked. The details can be found in [6] and Appendix C.) The clause C_0 is *already* redundant in $\exists X[F]$ in subspace $y_0 = 1$. So, adding K'' to F is optional.

By resolving $K' = y_0$ and $K'' = \overline{y}_0 \lor \overline{x}_2$ one obtains the certificate $K = \overline{x}_2$ that implies the target clause $C_0 = \overline{x}_2 \lor x_3$ in the entire space. This proves that C_0 is redundant in the current formula $\exists X[F]$ globally. Recall that currently H = K' and $F = K' \land F_{init}$ where F_{init} is the initial formula F. Since H is implied by F_{init} and adding H makes C_0 redundant in $H \land \exists X[F_{init}]$, both conditions of Proposition 2 are met. Hence $H = K' = y_0$ is a solution to our PQE problem i.e., $\exists X[F_{init}] \equiv y_0 \land \exists X[F_{init} \setminus \{C_0\}]$.

3.4 An example of changing the target clause

Let $F = C_0 \wedge C_1 \wedge C_2 \wedge \ldots$ where $C_0 = y_0 \vee x_1$, $C_1 = \overline{x}_1 \vee x_2 \vee x_3$, $C_2 = \overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3$. Let C_1 and C_2 be the only clauses of F with the literal \overline{x}_1 . Consider the problem of taking C_0 out of $\exists X[F(X,Y)]$ (we assume that $y_0 \in Y$ and $x_1, x_2, x_3 \in X$). Consider the subspace $\vec{q} = (y_0 = 0)$. In this subspace, the target clause C_0 turns into the unit clause x_1 . So, *START* changes the target C_0 and switches to proving redundancy of C_1 and C_2 in subspace \vec{q} . That is C_1 and then C_2 consecutively become the target clause in subspace \vec{q} .

Assume that certificates $K_1 = y_0 \vee x_2$ and $K_2 = y_0 \vee \overline{x}_3$ are derived for C_1 and C_2 respectively asserting their redundancy in subspace \vec{q} . Then C_0 is blocked in subspace \vec{q} and the certificate $K_0 = y_0 \vee x_1$ is derived as described above (and Appendix C). K_0 states the redundancy of C_0 in subspace \vec{q} .

Now assume that C_1 or C_2 is not redundant in subspace \vec{q} . This is possible only if F is unsatisfiable in subspace \vec{q} . In this case, the certificate $K_0 = y_0$ is derived. If K_0 is obtained using C_0 , it is added to F. (Otherwise, adding K_0 is optional.) K_0 asserts the redundancy of C_0 in subspace \vec{q} .

4 Property Generation And Local Computing

In this section, we discuss property generation by PQE [4] in the context of local computing. We mostly describe property generation for combinational circuits but in Subsection 4.5 we consider the case of sequential circuits. Property generation nicely illustrates the fact that PQE can be dramatically more efficient than QE. The complexity of PQE can reduce even to linear (see Subsection 4.4.)

4.1 Motivation for property generation and some background

Roughly speaking, hardware design verification consists of two steps. First, a set of specification properties is verified by formal tools [17]. Due to incompleteness of specification, even if those properties hold for an implementation Impl, the latter can still be buggy. Second, to address the problem above, some testing (simulation) procedures are applied to Impl [18,19,20]. Testing can be viewed as an example of *local computing* where only a tiny part of the search space is used. As we show in Subsection 4.3, one can represent the input/output behavior of Impl corresponding to a single test as a simple property. So, one can view the second step as generation of simple properties aimed at producing an *unwanted* one. Finding an unwanted property means that Impl is buggy. Since testing checks only simple properties, it can overlook a more complex unwanted property of Impl and hence miss a bug. The property generation procedure by PQE described below can produce non-trivial unwanted properties. So it can be viewed as a *generalization* of testing.

4.2 Property generation and local computing

Many design properties are inherently local for two reasons. First, some properties reflect the functionality of a small part of the design. These properties are **structurally-local**. Second, some properties like tests relate to a particular part of the functional space. Such properties are **functionally-local**. One can use PQE to generate properties of both types.

4.3 Testing as property generation

In this subsection, we use combinational circuits to relate tests and properties. Let M(X, V, W) be a combinational circuit where X, V, W specify the set of the internal, input, and output variables of M respectively. Let F(X, V, W) denote a formula specifying M. As usual, this formula is obtained by Tseitsin's transformations [21]. Namely, $F = F_{g_0} \wedge \cdots \wedge F_{g_k}$ where g_0, \ldots, g_k are the gates of M and F_{g_i} specifies the functionality of gate g_i .

Example 2. Let g be a 2-input AND gate defined as $x_2 = x_0 \wedge x_1$ where x_2 denotes the output value and x_0, x_1 denote the input values of g. Then g is specified by the formula $F_g = (\overline{x}_0 \vee \overline{x}_1 \vee x_2) \wedge (x_0 \vee \overline{x}_2) \wedge (x_1 \vee \overline{x}_2)$. Every clause of F_g is falsified by an inconsistent assignment (where the output value of g is not implied by its input values). For instance, $x_0 \vee \overline{x}_2$ is falsified by the inconsistent assignment $x_0 = 0, x_2 = 1$. So, every assignment satisfying F_g corresponds to a consistent assignment to g and vice versa. Similarly, every assignment satisfying the formula F above is a consistent assignment to the gates of M and vice versa.

The truth table T(V, W) of M can be obtained by QE, namely, $T \equiv \exists X[F]$. The formula T specifies the **strongest property** of M. However, computing T for a large circuit is, in general, infeasible. (This is the case, for instance, if M is obtained by unrolling a sequential circuit for k time frames.) Then one can verify M by running single tests. Let \vec{v} be a test i.e., a full assignment to V. Let \vec{w} be the output produced by M under the input \vec{v} . Denote by $H_{\vec{v}}(V,W)$ the formula such that $H_{\vec{v}}(\vec{v}',\vec{w}') = 0$ only if $\vec{v}' = \vec{v}$ and $\vec{w}' \neq \vec{w}$. (Otherwise, $H_{\vec{v}}(\vec{v}',\vec{w}') = 1$.) Formula $H_{\vec{v}}$ is implied by F and so is a property of M. This property specifies the input/output behavior of M for a single test (i.e., the test \vec{v}). We will refer to $H_{\vec{v}}$ as a **single-test property**. If the value \vec{w} produced by M under the input \vec{v} is wrong, the property $H_{\vec{v}}$ above is *unwanted* and M is buggy. Single-test properties are the **weakest properties** of M.

4.4 Property generation by PQE

Consider the PQE problem of taking a set of clauses G out of $\exists X[F]$. Let H(V, W) be a solution, i.e., $\exists X[F] \equiv H \land \exists X[F \setminus G]$. Since $F \Rightarrow H$, the solution H is a **property** of M. If H is an **unwanted** property, M has a bug. (Every subset of H specifies a property of M too. This property can be unwanted as well.) In general, to produce a property of M one can also quantify some input/output variables. For instance, by taking G out of $\exists X \exists V[F]$ one obtains a property H(W) depending only on output variables. This property asserts that M cannot produce an output falsifying H. Combining PQE with clause splitting described below one can generate properties of M that, in terms of strength, range from single-test properties to the truth table T above.

Using PQE for property generation is beneficial in three aspects. First, by taking out G one can produce a **structurally-local** property related to the part of the design specified by G. Second, if G is small, the property H can be computed much more efficiently than the truth table T obtained by QE. Third, by taking out different subsets of F one produces properties relating to different parts of the design. So one can estimate the completeness of property generation using some design coverage metric like it is done in testing. The intuition here is that if G relates to a buggy part, one is likely to produce an *unwanted* property identifying the bug. So, having a coverage metric helps to detect more bugs.

One can use *clause splitting* to compute **functionally-local** properties. Here we consider clause splitting on (some) input variables $v_0, \ldots, v_p \in V$ but one can split a clause on any subset of variables from Vars(F). Let H be a property obtained by taking out a single clause C i.e., $G = \{C\}$. Let F' denote the formula

F where <u>C</u> is replaced with the following p + 2 clauses: $C_0 = C \vee \overline{l(v_0)}, \ldots, C_p = C \vee \overline{l(v_p)}, C_{p+1} = C \vee l(v_0) \vee \cdots \vee l(v_p)$, where $l(v_i)$ is a literal of v_i . The idea is to obtain a weaker property H' by taking the clause C_{p+1} out of $\exists X[F']$ rather than C out of $\exists X[F]$. The formula H' describes a property related to the subspace falsifying the literals $l(v_0), \ldots, l(v_p)$. (So this property can be viewed as functionally-local.) One can show [4] that if $\{v_0, \ldots, v_p\} = V$ i.e., C is split on all input variables, then a) PQE has **linear** complexity; b) taking out C_{p+1} produces a **single-test property** $H_{\vec{v}}$ corresponding to the test \vec{v} falsifying the literals $l(v_0), \ldots, l(v_p)$.

4.5 Property generation for sequential circuits

In this subsection, we show how property generation for a combinational circuit can be extended to a sequential circuit N. Let $M_k(X, V, W)$ be the combinational circuit obtained by unrolling N for k transitions. Here X and V denote the internal and input variables of k time frames of N respectively and W denotes the *next state* variables of k-th time frame. In other words, X, V, W specify the internal, input and output variables of M_k . Let F_k be a formula specifying the circuit M_k and $H_k(W)$ be a property obtained by taking a clause of C out of $\exists X \exists V[F_k]$. That is $\exists X \exists V[F_k] \equiv H_k \land \exists X \exists V[F_k \setminus \{C\}]$.

The idea here is to check if separate clauses of H_k are invariants of N. (Even if H_k itself is not an invariant, some clauses of H_k may be.) Let Q be a clause of H_k . This clause specifies the property of M_k stating that it cannot produce an output \vec{w} falsifying Q. In terms of N this means that no state \vec{w} falsifying Q can be reached in k transitions. To check if Q is an invariant, one can run a model checker to see if every reachable state of N satisfies Q.

Assume Q holds. This means that any state falsifying Q is unreachable by N. If Q is falsified by a state that is *supposed to be reachable* by N, then Q is an *unwanted invariant* and N has a bug. Now assume that Q is supposed to hold but it *fails*. This means that N has an *unwanted property* \overline{Q} and hence is buggy.

4.6 Experimental results

In this subsection, to give an idea about the status quo, we describe some experimental results on property generation reported in [4]. Those results were obtained by the PQE solver called $EG-PQE^+$ that was introduced in [4]. $EG-PQE^+$ was used to generate properties for the combinational circuit M_k obtained by unfolding a sequential circuit N for k time frames. Those properties were employed to generate invariants of N. A sample of HWMCC benchmarks containing from 100 to 8,000 latches was used in those experiments. With the time limit of 10 seconds, $EG-PQE^+$ managed to generate a lot of properties of M_k that turned out to be invariants of N. $EG-PQE^+$ also successfully generated an unwanted invariant of a tailor-made FIFO buffer and so identified a hard-to-find bug.

5 Equivalence Checking And Local Computing

In this section, we discuss equivalence checking by PQE [3] in the context of local computing. Although this discussion is limited to combinational circuits, the method introduced in [3] can be extended to equivalence checking of more complex entities e.g. sequential circuits.

5.1 Motivation and some background

Equivalence checking is of great importance for two reasons. First, it is used in design verification for proving that a modified circuit remains equivalent to some golden reference model. Second, logic synthesis is only as powerful as equivalence checking: when optimizing a circuit, only logic transformations that can be efficiently checked for equivalence are used. So, making equivalence checking more powerful has a profound effect on design quality.

In general, equivalence checking is hard even for combinational circuits. However, in practice, one usually deals with *structurally similar* circuits [22,23,24]. In this case, equivalence checking can be very efficient. The best methods here employ computing simple predefined relations (e.g., equivalences) between internal points of the pair of circuits to compare [25,26,27]. Unfortunately, two circuits can be structurally similar even if they have no internal points related by predefined relations (or have very few of them). The method of equivalence checking based on PQE [3] that we recall in Subsection 5.3 solves this problem. This method can exploit the similarity of the circuits to compare without looking for any predefined relations between internal points of these circuits.

5.2 Equivalence checking and local computing

Let M'(X', V', w') and M''(X'', V'', w'') be the single-output combinational circuits to check for equivalence. Here X^{α}, V^{α} are the sets of internal and input variables and w^{α} is the output variable of M^{α} where $\alpha \in \{', ''\}$. Circuits M', M'' are called **equivalent** if they produce the same output for every full assignment

to V. Intuitively, if M', M'' are structurally similar, their equivalence can be established **locally** due to the existence of simple relations between corresponding internal points of M', M''. So one can derive these relations in an induction-like manner from inputs to outputs until the equivalence of w' and w'' is proved.

Figure 1 illustrates a method capturing the intuition above. We will refer to it as the *CP* method where "CP" stands for "Cut Propagation". (Modern equivalence checkers use a version of *CP* and so can be viewed as an example of *local computing*. However, their version of *CP* is very limited, see below.) The *CP* method creates a series of cuts Cut_0, \ldots, Cut_k in M', M'' where $Cut_0 = V' \cup V''$ and $Cut_k = \{w', w''\}$. For each cut Cut_i , $0 < i \le k$, the *CP* method computes a formula Rel_i relating variables of Cut_i . For Cut_0 , the formula Rel_0 equals EQ(V', V'') where $EQ(\vec{v}', \vec{v}'') = 1$ iff $\vec{v}' = \vec{v}''$. Here \vec{v}', \vec{v}'' are full assignments to V' and V'' respectively. (*EQ* forces one to compare M'and M'' only for identical assignments to V' and V''.) Rel_i is computed using the relations of the previous cuts Rel_j , $0 \le j < i$. The *CP* method eventually derives Rel_k for the last cut $Cut_k = \{w', w''\}$. If $Rel_k \Rightarrow (w' \equiv w'')$, then M'and M'' are equivalent.

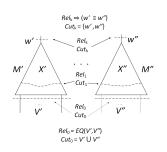


Fig. 1: Proving equivalence by the CP method

The CP method above does not explain when to stop computing Rel_i and move to building Rel_{i+1} . So, in practice, to implement the CP method, the set of possible relations in Rel_i is dramatically reduced, typically, to the functional equivalence of cut points of M'and M''. As mentioned above, equivalent cut points may be scarce or not exist at all even if M', M'' are very similar. In that case, the limited version of the CP method above cannot prove M', M'' equivalent.

5.3 Equivalence checking by PQE

In this subsection, we describe a variation of the *CP* method called CP^{pqe} where relations Rel_i are computed by PQE. In contrast to *CP*, the CP^{pqe} method describes how Rel_i are constructed. So, CP^{pqe} can be applied to any pair of circuits M', M'' but is especially efficient if M'and M'' are structurally similar. Below, we assume that neither M' nor M'' is a constant, which can be verified by a few easy SAT-checks. (To show, say, that M' is not a constant one just needs to check if M' can output both 0 and 1.)

Let $F = F' \wedge F''$ where F'(X', V', w') and F''(X'', V'', w'') specify M' and M'' respectively as described in Subsection 4.3. Let $Z = X' \cup X'' \cup V' \cup V''$. A straightforward (but hugely inefficient) method of equivalence checking is to perform QE on $\exists Z[F]$. Let $H(w', w'') \equiv \exists Z[EQ \wedge F]$. Circuits M', M'' are equivalent iff $H \Rightarrow (w' \equiv w'')$. The CP^{pqe} method is based on the observation that H(w', w'') can be computed by taking EQ out of $\exists Z[EQ \wedge F]$ i.e., by PQE. **Proposition 3.** Assume that M', M'' are not constants. Assume that $\exists Z[EQ \land F] \equiv H \land \exists Z[F]$. Then M' and M'' are equivalent iff $H \Rightarrow (w' \equiv w'')$.

The CP^{pqe} method solves this PQE problem *incrementally* by building a sequence of cuts. Like in the CP method, for each cut Cut_i , $0 < i \leq k$, the CP^{pqe} method computes Rel_i . As before, $Rel_0 = EQ$. The formula Rel_1 is obtained by PQE [3], see Appendix D. It has the property $\exists Z[Rel_0 \wedge F] \equiv \exists Z[Rel_1 \wedge F]$. (In other words, adding Rel_1 makes Rel_0 redundant in $\exists Z[Rel_0 \wedge Rel_1 \wedge F]$.) The CP^{pqe} method uses this fact to replace Rel_0 with Rel_1 . Similarly, adding Rel_i , $1 < i \leq k$ makes Rel_{i-1} redundant in $\exists Z[Rel_{i-1} \wedge Rel_i \wedge F]$, so Rel_i replaces Rel_{i-1} . Eventually, CP^{pqe} produces $Rel_k(w', w'')$ such that $\exists Z[EQ \wedge F] \equiv \exists Z[Rel_0 \wedge F] \equiv \exists Z[Rel_1 \wedge F] \equiv \cdots \equiv \exists Z[Rel_k \wedge F] \equiv Rel_k \wedge \exists Z[F]$. In other words, Rel_k is a solution to the PQE problem of taking EQ out of $\exists Z[EQ \wedge F]$. The circuits M' and M'' are equivalent iff $Rel_k \Rightarrow (w' \equiv w'')$.

5.4 CP^{pqe} and local computing

 CP^{pqe} facilitates exploiting the inherent locality of equivalence checking when M' and M'' are structurally similar. First, CP^{pqe} clearly identifies the moment when constructing $Rel_{i,i} > 0$ is over. Namely, this occurs when Rel_{i-1} becomes redundant in $\exists Z[Rel_{i-1} \land Rel_i \land F]$. (After that CP^{pqe} starts building Rel_{i+1} .) Second, Rel_i itself consists of a set of short (i.e., "local") clauses. This claim is substantiated by the following result [3]. Consider the form of similarity of M' and M'' where there is a small number p such that every cut point of Cut_i of M' (respectively M'). Then there is Rel_i consisting of clauses with no more than p+1 literals that makes Rel_{i-1} redundant in $\exists X[Rel_{i-1} \land Rel_i \land F]$. So, CP^{pqe} is able to exploit the similarity of M' and M'' via generating Rel_i consisting of short clauses. The version of the CP method (used in the current commercial tools) that looks for functionally equivalent cut points of M', M'' is just a **special case** of the similarity above where p=1.

5.5 Experimental results

An experiment with an implementation of CP^{pqe} was presented in [3]. In that experiment, pairs of M', M'' containing a multiplier of various sizes were checked for equivalence. (The size of the multiplier ranged from 10 to 16 bits.) M', M''were intentionally designed so that they were structurally similar but did not have any functionally equivalent points. A high-quality tool called ABC [28] showed very poor performance whereas CP^{pqe} solved all examples efficiently. In particular, CP^{pqe} solved the example involving a 16-bit multiplier in 70 seconds whereas ABC failed to finish it in 6 hours.

6 Model Checking And Local Computing

In this section, we consider finding the reachability diameter, which is a problem of model checking. To solve this problem we use local computing by PQE.

6.1 Motivation and some background

An obvious application for an efficient algorithm for finding the reachability diameter is as follows. Suppose one knows that the reachability diameter of a sequential circuit N is less or equal to k. Then to verify any invariant of N it suffices to check if it holds for the states of N reachable in at most k transitions. This check can be done by bounded model checking [29]. Finding the reachability diameter of a sequential circuit by existing methods essentially requires computing the set of all reachable states [30,31], which does not scale well. An upper bound on the reachability diameter called the recurrence diameter can be found by a SAT-solver [32]. However, this upper bound is very imprecise. Besides, its computing does not scale well either.

6.2 Some definitions

Let T(S', S'') denote the transition relation of a sequential circuit N where S', S'' are the sets of present and next state variables. Let formula I(S) specify the initial states of N. (A **state** is a full assignment to the set of state variables.) A state \vec{s}_k of N with initial states I is called **reachable** in k transitions if there is a sequence of states $\vec{s}_0, \ldots, \vec{s}_k$ such that $I(\vec{s}_0) = 1$ and $T(\vec{s}_{i-1}, \vec{s}_i) = 1$, $i = 1, \ldots, k$. For the reason described in Remark 1, we assume that N can **stutter** that is, $T(\vec{s}, \vec{s}) = 1$ for every state \vec{s} . (If N lacks stuttering, the latter can be easily introduced.)

Remark 1. If N can stutter, its set of reachable states is easier to describe because a state of N reachable in p transitions is also reachable in k transitions where k > p. So, the set of states of N reachable in k transitions is the same as the set of states reachable in at most k transitions.

Let R_k be a formula specifying the set of states of N reachable in k transitions. That is $R_k(\vec{s}) = 1$ iff \vec{s} is reachable in k transitions. Formula $R_k(S_k)$ can be computed by performing QE on $\exists S_{0,k-1}[I_0 \wedge T_{0,k-1}]$ where $S_{0,k-1} = S_0 \cup \cdots \cup S_{k-1}$ and $T_{0,k-1} = T(S_0, S_1) \wedge \cdots \wedge T(S_{k-1}, S_k)$. We will call Diam(I, N) the **reachability diameter** of N with initial states I if any reachable state of N requires at most Diam(I, N) transitions to reach it.

6.3 Computing reachability diameter and local computing

In this subsection, we consider the problem of deciding if $Diam(I, N) \leq k$. A straightforward way to solve this problem is to compute R_k and R_{k+1} by performing QE as described above. $Diam(I, N) \leq k$ iff R_k and R_{k+1} are equivalent. Unfortunately, computing R_k , R_{k+1} even for a relatively small value of k can be very hard or simply infeasible for large circuits. One can address this issue by exploiting the **locality** of the problem. First, R_k, R_{k+1} are different only in one transition, so they are, in a sense, *close*. Second, checking if $R_k \equiv R_{k+1}$ holds can be done *without* computing R_k and R_{k+1} explicitly, i.e., "globally".

Proposition 4. Let $k \ge 0$. Let $\exists S_{0,k}[I_0 \land I_1 \land T_{0,k}]$ be a formula where I_0 and I_1 specify the initial states of N in terms of variables of S_0 and S_1 respectively, $S_{0,k} = S_0 \cup \cdots \cup S_k$ and $T_{0,k} = T(S_0, S_1) \land \cdots \land T(S_k, S_{k+1})$. Then $Diam(I, N) \le k$ iff I_1 is redundant in $\exists S_{0,k}[I_0 \land I_1 \land T_{0,k}]$.

Proposition 4 reduces checking if $Diam(I, N) \leq k$ to finding if I_1 is redundant in $\exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$ (which is the *decision version* of PQE). Importantly, I_1 is a small piece of the formula. So, proving it redundant can be much more efficient than computing R_k and R_{k+1} . (Computing R_{k+1} by QE requires, for instance, proving the *entire formula* $I_0 \wedge T_{0,k}$ redundant in $R_{k+1} \wedge \exists S_{0,k}[I_0 \wedge T_{0,k}]$.)

7 SAT And Local Computing

In this section, we discuss solving the satisfiability problem (SAT) by PQE in the context of local computing. Given a formula F(X), SAT is to check if F is satisfiable i.e., whether $\exists X[F]=1$.

7.1 Motivation, some background, and locality of SAT

Our interest in solving SAT by PQE is motivated by the tremendous role SAT plays in practical applications. Modern SAT solvers are descendants of the DPLL procedure [33] that checks the satisfiability of a formula F(X) by looking for a satisfying assignment. They identify subspaces where F is unsatisfiable and learn conflict clauses to avoid re-visiting those subspaces (see e.g. [1,34,35,36]).

There are at least three cases where SAT has some form of locality that cannot be fully exploited by the descendants of DPLL. The first case is that an assignment \vec{x} to X is known that could be close to an assignment satisfying F. For instance, \vec{x} satisfied F before the latter was modified. The second case is that F has a small unsatisfiable core, which is typical for real-like formulas. The third case occurs when a part of F becomes "unobservable" in a subspace i.e., redundant in $\exists X[F]$ in this subspace. Some global unobservabilities, i.e., those that hold in every subspace, can be removed by preprocessing procedures [37,38,39]. However, a real-life formula F can have a lot of *local* unobservabilities even after all global ones are gone. Suppose, for instance, that F specifies a circuit and the variable $x \in X$ describes an input of a 2-input AND gate g. Then, in any subspace where x = 0, the clauses of F specifying gates feeding the other input of the gate g could become unobservable.

7.2 Reducing SAT to PQE

Proposition 5 below reduces SAT to the "decision version" of QE formulated in terms of redundancy based reasoning. A traditional SAT algorithm solves this QE problem either by finding a satisfying assignment to prove F redundant in $\exists X[F]$ or by deriving an empty clause to prove otherwise.

Proposition 5. Formula F(X) is satisfiable iff F is redundant in $\exists X[F]$.

The proposition below reduces SAT to the decision version of PQE.

Proposition 6. Let F(X) be a formula and \vec{x} be a full assignment to X. Let G denote the set of clauses of F falsified by \vec{x} . Formula F is satisfiable iff G is redundant in $\exists X[F]$.

In terms of redundancy based reasoning, the PQE of Proposition 6 is easier than the QE of Proposition 5 in the following sense. In Proposition 5, one has to check if *all* clauses are redundant in $\exists X[F]$. Proposition 6 requires doing this only for the subset G of F. Moreover, one does not need to prove redundancy of G globally. It suffices to show that G is redundant in $\exists X[F]$ in some subspace \vec{q} where $\vec{q} \subseteq \vec{x}$. Then F is satisfiable in subspace \vec{q} .

7.3 Solving SAT by PQE and local computing

Let SAT^{pqe} be an algorithm solving SAT by using Proposition 6, i.e., by PQE. In this subsection, we briefly discuss why using an efficient SAT^{pqe} could be beneficial for solving the three cases of SAT mentioned in Subsection 7.1. First, assume that \vec{x} is close to an assignment satisfying F. The benefit of using SAT^{pqe} here is based on the following observation. Let C be a clause of G. The fact that x is close to a satisfying assignment makes it much easier to find a subspace $\vec{q} \subseteq \vec{x}$ where C is blocked and hence redundant in $\exists X[F]$. In Appendix E, we illustrate this observation by a simple example.

Now assume that F has an unsatisfiable core $F' \subset F$. Then at least one clause of F' is in G. (Otherwise, \vec{x} satisfies F'.) The smaller G, the higher the chance that an arbitrary clause of G is in F'. If G has only one clause, the latter is in F'. As we mentioned in Section 3, a PQE solver tries to prove redundancy of clauses that are structurally close. That is, if a target clause becomes unit, the PQE solver proves redundancy of clauses that are resolvable with this target clause. So, when proving the redundancy of a clause of $C \in G$ present in F', SAT^{pqe} has a natural trend to stay focused on clauses of F'. The formulas Fand G gradually change due to adding conflict clauses (i.e., conflict certificates). And to make G redundant, SAT^{pqe} eventually generates an empty clause.

Finally, assume that F has local unobservabilities. As we mentioned in Section 3, in addition to the conflict certificates of redundancy, a PQE solver also learns *non-conflict* certificates. The former identify the local inconsistencies of F (like in SAT solving) and the latter record its local *unobservabilites*. So, redundancy based reasoning employed by SAT^{pqe} provides a natural way to *locate* the part of F that matters, i.e., "observable", in the current subspace.

8 PQE And Interpolation

In this section, we show that interpolation [40,41] can be viewed as a special case of PQE. Our motivation here is twofold. First, one can think of interpolation as a form of local computing. Second, PQE looks similar to interpolation. So, it is natural to try to relate the two. Let $A(X, Y) \wedge B(Y, Z)$ be an unsatisfiable formula where X, Y, Z are sets of variables. Let I(Y) be a formula such that $A \wedge B \equiv I \wedge B$ and $A \Rightarrow I$. Replacing $A \wedge B$ with $I \wedge B$ is called *interpolation* and I is called an *interpolant*. Now, let us show that interpolation can be described in terms of PQE. Consider the formula $\exists W[A \wedge B]$ where $W = X \cup Z$ and A, B are the formulas above. Let $A^*(Y)$ be obtained by taking A out of the scope of quantifiers i.e., $\exists W[A \wedge B] \equiv A^* \wedge \exists W[B]$. Since $A \wedge B$ is unsatisfiable, $A^* \wedge B$ is unsatisfiable too. So, $A \wedge B \equiv A^* \wedge B$. If $A \Rightarrow A^*$, then A^* is an interpolant.

The general case of PQE that takes A out of $\exists W[A \land B]$ is different from the instance above in three aspects. First, $A \land B$ can be satisfiable. Second, one does not assume that $Vars(B) \subset Vars(A \land B)$. In other words, in general, PQE is not meant to remove variables of the original formula. Third, a solution A^* is, in general, implied by $A \land B$ rather than by A alone. Summarizing, one can say that interpolation is a special case of PQE.

9 Conclusions

We use the term local computing (LC) to refer to various techniques that reduce problem complexity by operating on a small piece of the formula or the search space. We relate LC to Partial Quantifier Elimination (PQE). The latter is a generalization of quantifier elimination where a *part* of the formula can be taken out of the scope of quantifiers. We apply LC by PQE to property generation, equivalence checking, model checking, SAT solving and interpolation.

LC by PQE allows to introduce a generalization of simulation (testing) called property generation where one identifies a bug by producing an *unwanted design property*. LC by PQE facilitates constructing an equivalence checker that exploits the similarity of the circuits to compare *without* searching for some predefined relations between internal points. In model checking, LC by PQE enables a procedure that finds the reachability diameter *without* computing the set of all reachable states. LC by PQE can potentially solve the SAT problems with various types of "locality" more efficiently. Finally, it can be shown that interpolation (a form of LC) is a special case of PQE. The results above suggest that studying PQE and designing fast PQE solvers is of *great importance*.

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Appendix

A Some Examples Of Local Computing

As we mentioned in the introduction, virtually all successful algorithms/techniques of hardware verification use some form of local computing. In this appendix, we give some examples of that.

A.1 Testing

Testing is a ubiquitous technique of functional verification. One of the reasons for such omnipresence is that the output produced by a circuit for a single test (i.e., an input assignment) can be efficiently computed. This efficiency, in turn, is due to the fact that a single test examines only a tiny part of the search space. So, testing can be viewed as an example of *local computing*.

A.2 Equivalence checking

Equivalence checking is one of the most efficient (and hence popular) techniques of formal verification. Let N' and N'' be the circuits to check for equivalence. In general, equivalence checking is a hard problem that does not scale well even if circuits N', N'' are combinational. Fortunately, in practice, N' and N'' are structurally similar. In this case, one can often prove the equivalence of N' and N'' quite efficiently by using local computing. The latter is to *locate* internal points of N' and N'' linked by simple relations like equivalence. These relations are propagated from inputs to outputs until the equivalence of the corresponding output variables of N' and N'' is proved. (If N' and N'' are sequential circuits, relations between internal points are propagated over multiple time frames.)

A.3 Model checking

A significant boost in model checking has been achieved due to the appearance of IC3 [42]. The idea of IC3 is as follows. Let N be a sequential circuit. Let P(S) be an invariant to prove where S is the set of state variables of N. (Proving P means showing that it holds in every reachable state of N.) IC3 looks for an *inductive* invariant P' such that $I \Rightarrow P' \Rightarrow P$ where I(S) specifies the initial states of N. IC3 builds P' by constraining P via adding so-called inductive clauses. The high scalability of IC3 can be attributed to the fact that in many cases P is "almost" inductive. So, to turn P into P', it suffices to add a relatively small number of clauses. Building P' as a variation of P can be viewed as a form of local computing.

A.4 SAT solving

The success of modern SAT-solvers can be attributed to two techniques. The first technique is the conflict analysis introduced by GRASP [1]. The idea is that when a conflict occurs in the current subspace, one identifies the set of clauses responsible for this conflict. This set is used to generate a so-called conflict clause that is falsified in the current subspace. So, adding it to the formula diverts the SAT algorithm from any subspace where the same conflict occurs. Since the set of clauses involved in a conflict is typically very small (in comparison to the entire formula), learning conflict clauses can be viewed as a form of *local computing*.

The second technique introduced by Chaff [34] is to employ decision making that involves variables of recent conflict clauses. The reason why Chaff-like decision making works so well can be explained as follows. Assume that a SAT-solver with conflict clause learning checks the satisfiability of a formula F. Assume that F is unsatisfiable. (If F is satisfiable, the reasoning below can be applied to every subspace visited by this SAT-solver where F was unsatisfiable.) Learning and adding conflict clauses produces an unsatisfiable core of F that includes learned clauses. This core gradually shrinks in size and eventually reduces to the core consisting of an empty clause. The decision making of Chaff helps to *locate* the subset of clauses/variables of F making up the current unsatisfiable core. So, one can view the second technique as a form of *local computing* as well.

B Proofs Of Propositions

Proposition 1. Let a clause C be blocked in a formula F(X,Y) with respect to a variable $x \in X$. Then C is redundant in $\exists X[F]$ i.e., $\exists X[F \setminus \{C\}] \equiv \exists X[F]$.

Proof. It was shown in [5] that adding a clause B(X) blocked in G(X) to the formula $\exists X[G]$ does not change the value of this formula. This entails that removing a clause B(X) blocked in G(X) does not change the value of $\exists X[G]$ either. So, B is redundant in $\exists X[G]$.

Let \vec{y} be a full assignment to Y. Then the clause C of the proposition at hand is either satisfied by \vec{y} or $C_{\vec{y}}$ is blocked in $F_{\vec{y}}$ with respect to x. (The latter follows from the definition of a blocked clause.) In either case, $C_{\vec{y}}$ is redundant in $\exists X[F_{\vec{y}}]$. Since this redundancy holds in every subspace \vec{y} , the clause C is redundant in $\exists X[F]$.

Proposition 2. Formula H(Y) is a solution to the PQE problem of taking a clause C out of $\exists X[F(X,Y)]$ (i.e., $\exists X[F] \equiv H \land \exists X[F \setminus \{C\}]$) iff

1. $F \Rightarrow H$ and 2. $H \land \exists X[F] \equiv H \land \exists X[F \setminus \{C\}]$

Proof. The if part. Assume that conditions 1, 2 hold. Let us show that $\exists X[F] \equiv H \land \exists X[F \setminus \{C\}]$. Assume the contrary i.e., there is a full assignment \vec{y} to Y such that $\exists X[F] \neq H \land \exists X[F \setminus \{C\}]$ in subspace \vec{y} .

There are two cases to consider here. First, assume that F is satisfiable and $H \wedge \exists X[F \setminus \{C\}]$ is unsatisfiable in subspace \vec{y} . Then there is an assignment (\vec{x}, \vec{y}) satisfying F (and hence satisfying $F \setminus \{C\}$). This means that (\vec{x}, \vec{y}) falsifies H and hence F does not imply H. So, we have a contradiction. Second, assume that F is unsatisfiable and $H \wedge \exists X[F \setminus \{C\}]$ is satisfiable in subspace \vec{y} . Then $H \wedge \exists X[F]$ is unsatisfiable too. So, condition 2 does not hold and we have a contradiction.

The only if part. Assume that $\exists X[F] \equiv H \land \exists X[F \setminus \{C\}]$. Let us show that conditions 1 and 2 hold. Assume that condition 1 fails i.e., $F \not\Rightarrow H$. Then there is an assignment (\vec{x}, \vec{y}) satisfying F and falsifying H. This means that $\exists X[F] \neq H \land \exists X[F \setminus \{C\}]$ in subspace \vec{y} and we have a contradiction. To prove that condition 2 holds, one can simply multiply both sides of the equality $\exists X[F] \equiv H \land \exists X[F \setminus \{C\}]$ by H.

Proposition 3. Assume that M', M'' are not constants. Assume that $\exists Z[EQ \land F] \equiv H \land \exists Z[F]$. Then M' and M'' are equivalent iff $H \Rightarrow (w' \equiv w'')$.

Proof. The if part. Assume that $H \Rightarrow (w' \equiv w'')$. From Proposition 2 it follows that $(EQ \land F) \Rightarrow H$. So $(EQ \land F) \Rightarrow (w' \equiv w'')$. Recall that $F = F' \land F''$ where F' and F'' specify M' and M'' respectively. So, for every pair of inputs \vec{v}' and \vec{v}'' satisfying EQ(V', V'') (i.e., $\vec{v}' = \vec{v}''$), M' and M'' produce identical values of w' and w''. Hence, M' and M'' are equivalent.

The only if part. Assume the contrary i.e., M' and M'' are equivalent but $H(w', w'') \not\Rightarrow (w' \equiv w'')$. There are two possibilities here: H(0, 1) = 1 or H(1, 0) = 1. Consider, for instance, the first possibility i.e., w' = 0, w'' = 1. Since, M' and M'' are not constants, there is an input \vec{v}' for which M' outputs 0 and an input \vec{v}'' for which M'' outputs 1. This means that the formula $H \wedge \exists Z[F]$ is satisfiable in the subspace w' = 0, w'' = 1. Then the formula $\exists Z[EQ \wedge F]$ is satisfiable in this subspace too. This means that there is an input \vec{v} under which M' and M'' produce w' = 0 and w'' = 1. So, M' and M'' are inequivalent and we have a contradiction.

Proposition 4. Let $k \ge 0$. Let $\exists S_{0,k}[I_0 \land I_1 \land T_{0,k}]$ be a formula where I_0 and I_1 specify the initial states of N in terms of variables of S_0 and S_1 respectively, $S_{0,k} = S_0 \cup \cdots \cup S_k$ and $T_{0,k} = T(S_0, S_1) \land \cdots \land T(S_k, S_{k+1})$. Then $Diam(I, N) \le k$ iff I_1 is redundant in $\exists S_{0,k}[I_0 \land I_1 \land T_{0,k}]$.

Proof. The if part. Assume that I_1 is redundant in $\exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$ i.e., $\exists S_{0,k}[I_0 \wedge T_{0,k}] \equiv \exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$. The formula $\exists S_{0,k}[I_0 \wedge T_{0,k}]$ is logically equivalent to R_{k+1} specifying the set of states of N reachable in k+1 transitions. On the other hand, $\exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$ is logically equivalent to R_k (because I_0 and $T(S_0, S_1)$ are redundant in $\exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$). So, redundancy of I_1 means that R_k and R_{k+1} are logically equivalent and hence, $Diam(I, N) \leq k$.

The only if part. Assume the contrary i.e., $Diam(I, N) \leq k$ but I_1 is not redundant in $\exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$. Then there is an assignment $\vec{p} = (\vec{s}_0, \ldots, \vec{s}_{k+1})$ such that a) \vec{p} falsifies at least one clause of I_1 ; b) \vec{p} satisfies $I_0 \wedge T_{0,k}$; c) formula $I_0 \wedge I_1 \wedge T_{0,k}$ is unsatisfiable in subspace \vec{s}_{k+1} . (So, I_1 is not redundant because removing it from $I_0 \wedge I_1 \wedge T_{m+1}$ makes the latter satisfiable in subspace \vec{s}_{k+1} .) Condition c) means that \vec{s}_{k+1} is unreachable in k transitions whereas condition b) implies that \vec{s}_{k+1} is reachable in k + 1 transitions. Hence Diam(I, N) > k and we have a contradiction.

Proposition 5. Formula F(X) is satisfiable iff F is redundant in $\exists X[F]$.

Proof. The if part. Let F be redundant in $\exists X[F]$. Then $\exists X[F] \equiv \exists X[F \setminus F]$. Since an empty set of clauses is satisfiable, F is satisfiable too.

The only if part. Assume the contrary i.e., F is satisfiable and F is not redundant in $\exists X[F]$. This means that there is a formula H where $H \neq 1$ such that $\exists X[F] \equiv H \land \exists X[F \setminus F]$. Since all variables of F are quantified in $\exists X[F]$, then H is a constant. The only option here is H = 0. So, $\exists X[F] = 0$ and we have a contradiction.

Proposition 6. Let F(X) be a formula and \vec{x} be a full assignment to X. Let G denote the set of clauses of F falsified by \vec{x} . Formula F is satisfiable (i.e., $\exists X[F]=1$) iff the formula G is redundant in $\exists X[F]$.

Proof. The if part. Let G be redundant in $\exists X[F]$. Then $\exists X[F] \equiv \exists X[F \setminus G]$. Since \vec{x} satisfies $F \setminus G$, then $\exists X[F \setminus G] = 1$. Hence $\exists X[F] = 1$ too.

The only if part. Assume the contrary i.e., F is satisfiable and G is not redundant in $\exists X[F]$. This means that there is a formula H where $H \not\equiv 1$ such that $\exists X[F] \equiv H \land \exists X[F \setminus G]$. Since all variables of F are quantified in $\exists X[F]$, then H is a constant. The only option here is H = 0. So, $\exists X[F] = 0$ and we have a contradiction.

C Identification/Generation Of Certificates

In this appendix, we describe identification/generation of certificates by STARTin more detail. Consider the problem of taking a clause out of the scope of quantifiers in $\exists X[F]$. Let C be the current target clause. Let K be a certificate clause stating the redundancy of C in subspace \vec{q} . That is, K implies C in subspace \vec{q} . As we mentioned in Subsection 3.2, the certificate K has the property $\exists X[K \land F] \equiv \exists X[F]$. So, K can be safely added to F. The report [6] describes the four different ways to identify/generate K listed below.

- 1. There is a clause $K \in F$ that implies C in subspace \vec{q} . In this case, K itself serves as the certificate of redundancy of C in subspace \vec{q} . (No generation of a new certificate is needed.)
- 2. A conflict occurs in subspace \$\vec{q}\$ and a conflict clause \$K\$ is generated. If \$C\$ was used to generate \$K\$, the latter is added to \$F\$ to make \$C\$ redundant in subspace \$\vec{q}\$. Otherwise, \$C\$ is already redundant in subspace \$\vec{q}\$ and adding \$K\$ to \$F\$ is optional. In either case, \$K\$ serves as the certificate of redundancy of \$C\$ in subspace \$\vec{q}\$.
- 3. A new certificate K is obtained by resolving two certificates K' and K'' identified/generated earlier. Let K' and K'' state the redundancy of C in subspaces \vec{q}' and \vec{q}'' respectively. Let K', K'' be resolved on a variable v. The certificate K states the redundancy of C in the subspace \vec{q} equal to $\vec{q}' \cup \vec{q}''$ minus the assignments to v. Adding K to F is optional.
- 4. The clause C is blocked in subspace \vec{q} at a variable v. Let l(v) be the literal of v present in C. Then a certificate K of redundancy of C in subspace \vec{q} is built as follows. First, K contains l(v). Second, for every clause C' with the literal $\overline{l(v)}$, the certificate K contains literals of variables "responsible" for the fact that C' cannot be resolved with C on v in subspace \vec{q} . Namely,
 - if C' is satisfied by \$\vec{q}\$, the certificate K contains the negation of a literal of C' satisfied by \$\vec{q}\$;
 - if C and C' have opposite literals of more than one variable, K contains a literal l(w) (other than l(v)) such that C' contains $\overline{l(w)}$;
 - if C' is proved redundant in subspace \vec{q} with a certificate K', the certificate K contains all the literals of K' minus those present in C'.

Adding the certificate K to F is optional in item 4.

Example 3. Consider the problem of taking a clause out of $\exists X[F(X,Y)]$ were $F = C_0 \wedge C_1 \wedge C_2 \wedge C_3 \wedge \ldots$. Let $C_0 = x_1 \vee x_2 \vee x_3$ be the current target

clause and C_1, C_2, C_3 be the only clauses of F containing the literal \overline{x}_1 where $C_1 = y_0 \lor \overline{x}_1, C_2 = \overline{x}_1 \lor \overline{x}_3, C_3 = \overline{x}_1 \lor \overline{x}_5$. As usual, we assume here that $y_i \in Y$ and $x_i \in X$. Let $\overrightarrow{q} = (y_0 = 1, y_4 = 0, ...)$ be the current assignment. Assume that C_3 is proved redundant in subspace \overrightarrow{q} and this redundancy is asserted by the certificate $K' = y_4 \lor \overline{x}_5$.

The clause C_0 is blocked in subspace \vec{q} at x_1 because there is no clause resolvable with C_0 on x_1 in this subspace. Namely, C_1 is satisfied by \vec{q} , the clause C_2 and C_0 have the opposite literals of x_1 and x_3 , and C_3 is redundant in subspace \vec{q} . So, C_0 is redundant in subspace \vec{q} . This redundancy can be asserted by the certificate $K = \overline{y}_0 \lor y_4 \lor x_1 \lor x_3$. (As required, K implies C_0 in subspace \vec{q} .) The literal x_1 of C_0 is present in K because x_1 is the variable at which C_0 is blocked. The literal \overline{y}_0 is in K because the literal y_0 of C_1 is satisfied by \vec{q} . The literal x_3 is in K because C_0 and C_2 have the opposite literals of x_3 . Finally, the literal y_4 is in K because this literal is in the certificate K' asserting the redundancy of C_3 in subspace \vec{q} and it is not in C_3 .

D Computing Rel_i By CP^{pqe}

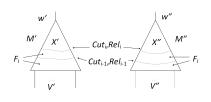


Fig. 2: Computing Rel_i in CP^{pqe}

In Section 5, we recalled CP^{pqe} , a method of equivalence checking by PQE introduced in [3]. In this appendix, we reuse the notation of Section 5 to describe how CP^{pqe} computes the formula Rel_i specifying relations between cut points of Cut_i . Let formula F_i specify the logic of M' and M'' located between their inputs and Cut_i (see Figure 2). Let Z_i denote the variables of F_i minus those of Cut_i . Then Rel_i is obtained by taking Rel_{i-1}

out of $\exists Z_i[Rel_{i-1} \wedge F_i]$ i.e., $\exists Z_i[Rel_{i-1} \wedge F_i] \equiv Rel_i \wedge \exists Z_i[F_i]$. The formula Rel_i depends only on variables of Cut_i . (All the other variables of $Rel_{i-1} \wedge F_i$ are in Z_i and hence, quantified.)

Note that since Rel_i is obtained by taking out Rel_{i-1} , the latter is redundant in $\exists Z_i[Rel_i \land Rel_{i-1} \land F_i]$. One can show that this implies redundancy of Rel_{i-1} in $\exists Z[Rel_{i-1} \land Rel_i \land F]$. Recall that F specifies the circuits M' and M'' and $Z = X' \cup X'' \cup V' \cup V''$. That is, Z includes all variables of F but the output variables w', w''. Then the property mentioned in Section 5 holds. Namely, $\exists Z[Rel_{i-1} \land F] \equiv \exists Z[Rel_i \land F]$.

E Simple Example Illustrating Observation Of Section 7

Let $\exists X[F(X)]$ specify the SAT problem to solve, \vec{x} be a full assignment to X and G be the set of clauses of F falsified by \vec{x} . Let \vec{x} be close to an assignment satisfying F and C be a clause of G. In Section 7 we mentioned the observation that the proximity of x to a satisfying assignment makes it easier to find a subspace $\vec{q} \subseteq \vec{x}$ where C is blocked in $\exists X[F]$. (Hence, it is redundant in $\exists X[F]$ in subspace \vec{q} .) Below, we illustrate this observation by a simple example³.

Example 4. Consider formula $\exists X[F]$ where $F = C_0 \wedge C_1 \wedge C_2 \wedge \ldots$ and $C_0 = x_0 \vee \overline{x}_1 \vee x_2$, $C_1 = \overline{x_0} \vee x_3$, $C_2 = \overline{x_0} \vee x_4$. Let the assignment \vec{x} above be equal to $(x_0 = 0, x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 1, \ldots)$. Note that C_0 is falsified while C_1, C_2 are satisfied by \vec{x} . Let C_0 be the only clause of F falsified by \vec{x} and C_1, C_2 be the only clauses of F with the literal \overline{x}_0 . This means that by flipping the value of x_0 one turns \vec{x} into an assignment satisfying F i.e. \vec{x} is very close to a satisfying assignment.

It is not hard to find a subset \vec{q} of \vec{x} such that C_0 is blocked in subspace \vec{q} . Consider for instance $\vec{q} = (x_3 = 1, x_4 = 1)$. Notice that C_0 is blocked in subspace \vec{q} at variable x_0 because C_1 and C_2 are satisfied by \vec{q} . So C_0 is redundant in subspace \vec{q} , which means that F is satisfiable. Finding the subspace $\vec{q} \subseteq \vec{x}$ where C_0 is blocked is so easy because \vec{x} is close to a satisfying assignment.

³ If \vec{x} is close to a satisfying assignment, a straightforward way to solve the SAT problem is to keep flipping values of \vec{x} until the latter satisfies F. This approach is used in so-called local search algorithms [43,44]. It works well for random formulas but fails on real-life ones. SAT^{pqe} provides a more powerful approach where one can learn conflict and non-conflict certificates thus pruning big chunks of space.