Quantifier Elimination by Dependency Sequents

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Outline

- Introduction
- Dependency sequents
- Algorithm description
- Experimental results
- Conclusions

Quantifier Elimination (QE)

Let F be a Boolean CNF formula and $X \subseteq Vars(F)$.

QE problem:

Given $\exists X\,[F\,]$, find a quantifier free CNF formula G such that $G \equiv \exists X\,[F\,]$

$G \equiv \exists X\,[F\,]$ means that $G_{\mathbf{s}} = \exists X\,[F_{\mathbf{s}}]$ for every complete assignment $\mathbf{s}$ to $Vars(F) \setminus X$

QE is important in reachability analysis and model checking

Existing QE Methods

BDD based methods suffer from memory explosion

Two kinds of SAT based QE algorithms:

- enumeration of satisfying assignments:

  McMillan 2002, Ganai,Gupta, Ashar 2004,

  Jin, Somenzi 2005, Brauer, King, Kriener 2011

- variable elimination:

  Davis, Putnam 1960, Jiang 2009, Goldberg,

  Manolios 2010

SAT based QE methods have poor scalability

Two Ideas of Our Approach

1) Add resolvent-clauses to F until variables of X become redundant in $\exists X\,[H]$ where $H \supseteq F$

   Redundancy means that $\exists X\,[H] \equiv \exists X\,[H \setminus H^X]$ where H^X = {the clauses of H with a variable of X}.

2) Use branching to prove variable redundancy in subspaces and merge results of different branches

Beyond Resolution

Termination condition: stop when X is redundant in $\exists X\,[F\,]$.

Let $Y = Vars(F) \setminus X$ and $\boldsymbol{s}$ be a complete assignment to Y.

$F_{\boldsymbol{s}}$ is unsatisfiable: there is a resolvent C , s.t. $C(\boldsymbol{s}) = 0$.
X is redundant in $\exists X\,[(F \wedge C)_{\boldsymbol{s}}]$. Resolution works.

$F_{\boldsymbol{s}}$ is satisfiable, there is no resolvent C s.t. $C(\boldsymbol{s}) = 0$.
Yet X is redundant in $\exists X\,[F_{\boldsymbol{s}}]$. Beyond resolution!

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Dependency Sequents (D-sequents)

Let $\boldsymbol{s}$ be a partial assignment to $Vars(F)$

A D-sequent for $\exists X\,[F\,]$ has the form

$(\exists X\,[F\,]\,,\,\boldsymbol{s}) \rightarrow\ Z,\quad where\ Z \subseteq X$

Semantics: Z is redundant in $\exists X\,[F_{\boldsymbol{s}}]$

We will call $\boldsymbol{s}$ the conditional part of the D-sequent

Properties:

a) If F implies G, $(\exists X\,[F \wedge G]\,,\,\boldsymbol{s}\,) \rightarrow\ Z$ also holds

b) If $\boldsymbol{s} \subseteq \boldsymbol{q}$, $(\exists X\,[F\,]\,,\,\boldsymbol{q}\,) \rightarrow\ Z$ also holds

Atomic D-sequents

1. Let $v \in X$ be monotone in F in subspace $\boldsymbol{s}$ (pure literal).

   Then v is redundant in $\exists X\,[F_{\boldsymbol{s}}]$ and so

   D-sequent $(\exists X\,[F], \boldsymbol{s}) \rightarrow \{v\}$ holds.

2. Let a clause C of F be falsified in subspace $\boldsymbol{s}$

   Every $v \in X \setminus Vars(\boldsymbol{s})$ is redundant in $\exists X\,[F_{\boldsymbol{s}}]$

   So D-sequent $(\exists X\,[F], \boldsymbol{s}) \rightarrow \{v\}$ holds

Often, $(\exists X\,[F], \boldsymbol{s}) \rightarrow \{v\}$ can be replaced with $(\exists X\,[F], \boldsymbol{s^*}) \rightarrow \{v\}$ where $\boldsymbol{s^*} \subset \boldsymbol{s}$

Joining D-sequents

$(\exists X\,[F\,], \boldsymbol{s}') \rightarrow Z$

where $\boldsymbol{s}' = (v_1{=}0, v_2{=}1, v_3{=}0)$

$(\exists X\,[F\,], \boldsymbol{s}'') \rightarrow Z$

where $\boldsymbol{s}'' = (v_1{=}1, v_2{=}1, v_{10}{=}0)$

$(\exists X\,[F\,], \boldsymbol{s}) \rightarrow Z$

where $\boldsymbol{s} = (v_2{=}1, v_3{=}0, v_{10}{=}0)$

Joining D-sequents at v_1

Importantly, $F_{\boldsymbol{s}'}$ and/or $F_{\boldsymbol{s}''}$ may be satisfiable.
Here we go beyond resolution that reasons only over subspaces where F is unsatisfiable.

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Derivation of D-sequents (DDS)

Ω is the current set of D-sequents

$(\exists X\,[F\,], \mathbf{s}_v) \rightarrow \{v\}$, where $v \in X$

DDS($\exists X\,[F\,]$, $\mathbf{s}$, Ω) /* Returns $\exists X\,[F\,]$, Ω, C */

1. (Ω, C) := *atomic_D_seqs*($\exists X\,[F\,]$, $\mathbf{s}$, Ω)
2. if (vars of X are assign. or redund.) return ($\exists X\,[F]$,Ω,C)
3. v:= *pick_var* (*Vars*(F) \ (*Assigned*($\mathbf{s}$) $\cup$ *Redundant*(Ω))
4. ($\exists X\,[F\,]$,Ω,C_0) := *DDS* ($\exists X\,[F\,]$,$\mathbf{s} \cup (v=0)$, Ω)
5. ($\exists X\,[F\,]$,Ω,C_1) := *DDS* ($\exists X\,[F\,]$, $\mathbf{s} \cup (v=1)$, $\Omega \setminus \Omega_{(v=0)}$)
6. Return (*Merge_Branches*($\exists X\,[F\,]$,$\mathbf{s}$,v,$\Omega_{(v=0)}$,Ω,C_0,C_1))

Merging Branches

Merge_Branches($\exists X\ [F\ ]$,$\boldsymbol{s}$,v,$\Omega_{(v=0)}$,Ω,C_0,C_1)

1. if (($C_0 \neq nil$) and ($C_1 \neq nil$))
2. { C := *resolve*(C_0,C_1,v)
3. $F := F \wedge C$
4. Ω := *update_D_seqs*($\boldsymbol{s}$,C,Ω)
5. return($\exists X\ [F\ ]$, Ω,C) }
6. Ω := *join_D_seqs*(v,$\Omega_{(v=0)}$,$\Omega_{(v=1)}$)
7. if ($v \in X$)
8. $\Omega := \Omega \cup$ { *atomic_D_seq*($\exists X\ [F\ ]$,$\boldsymbol{s}$,v) }
9. return($\exists X\ [F\ ]$,Ω,nil)

A Few Remarks About DDS

- To simplify implementation of DDS
  a) it first branches on variables of $Vars(F) \setminus X$
  b) D-sequents are not re-used

- DDS first branches on vars of unit clauses (BCP)

- If F is unsatisfiable, DDS behaves as a conflict driven SAT-solver. (But this is a “degenerate” case.)
- The novelty of DDS comes into play in subspaces where F is satisfiable.

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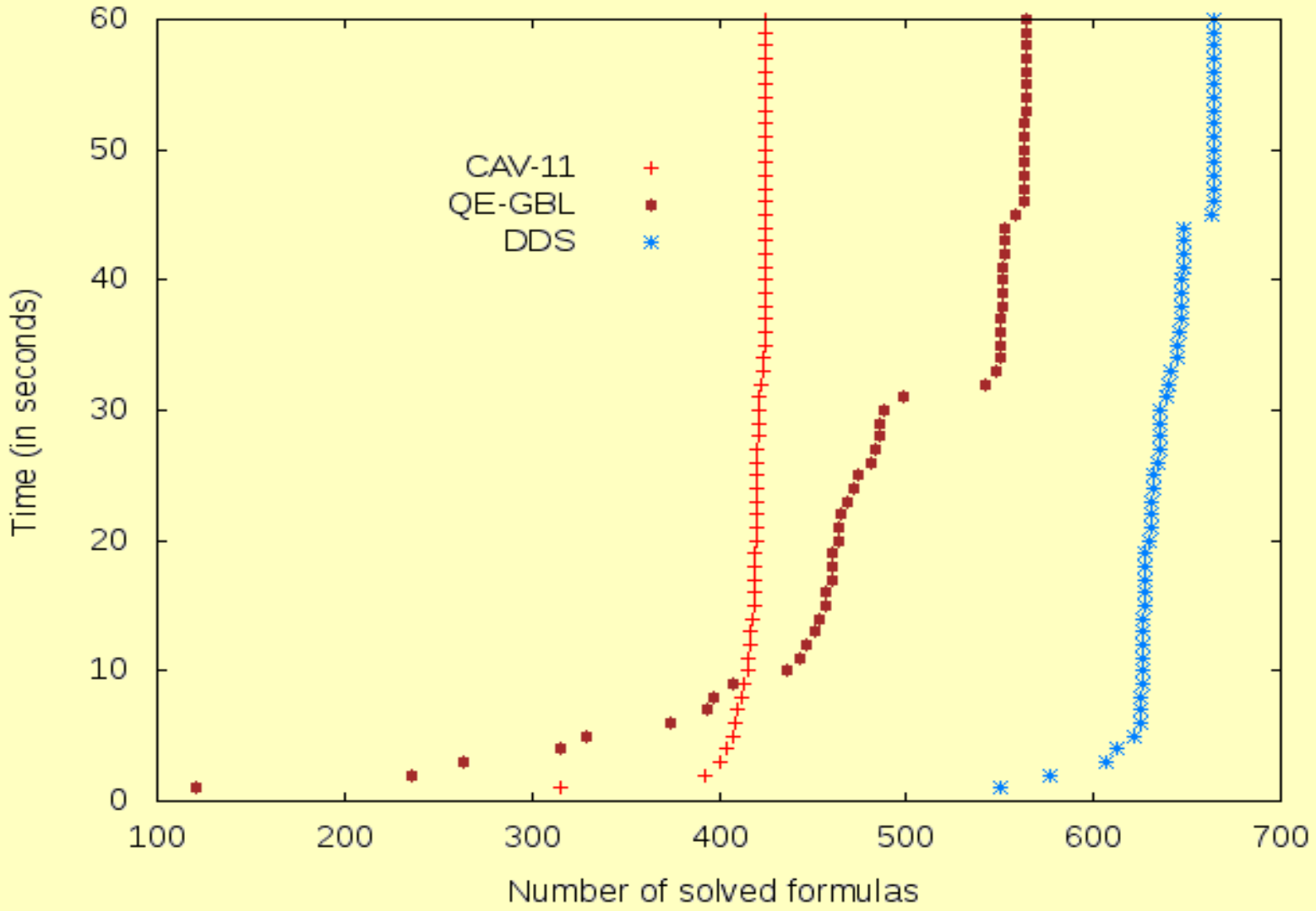
Model Checking Experiments

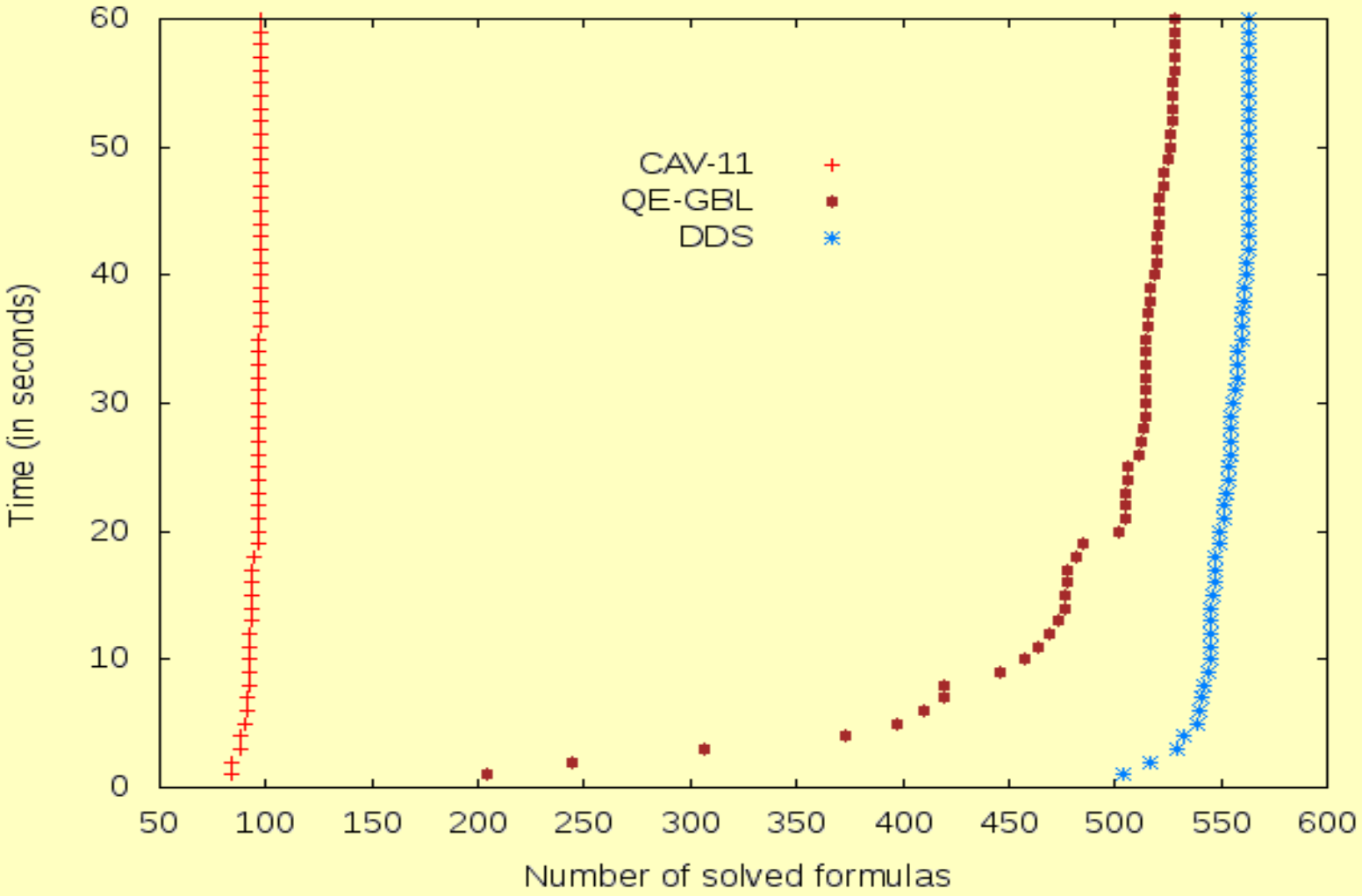
Model checking mode	**EnumSA** solved (%)	**QE-GBL** solved (%)	**DDS** solved (%)
forward	425 (56%)	561 (74%)	664 (87%)
backward	97 (12%)	522 (68%)	563 (74%)

- **EnumSA**: enumeration of satisf. assignments (CAV-11)
- **QE-GBL :** quantifies variables away globally (HVC-10)
- 758 benchmarks from HWMCC'10
- The algorithms performed one step of forward/backward model checking
- Timeout limit is 1 minute

Forward Model Checking



Backward Model Checking



Conclusions

- We introduced a QE algorithm based on D-sequents
- D-sequents can be used in many other applications
- We experimented with a very simple implementation
  The results of experiments are very encouraging
- Some points not covered in this talk are addressed in the paper (e.g. the compositionality of DDS)