## Quantifier Elimination With Structural Learning

Eugene Goldberg

eu.goldberg@gmail.com

Abstract. We consider the Quantifier Elimination (QE) problem for propositional CNF formulas with existential quantifiers. QE plays a key role in formal verification. Earlier, we presented an approach based on the following observation. To perform QE, one just needs to add a set of clauses depending on free variables that makes the quantified clauses (i.e. clauses with quantified variables) redundant. To implement this approach, we introduced a branching algorithm making quantified clauses redundant in subspaces and merging the results of branches. To implement this algorithm we developed the machinery of D-sequents. A D-sequent is a record stating that a quantified clause is redundant in a specified subspace. Redundancy of a clause is a structural property (i.e. it holds only for a subset of logically equivalent formulas as opposed to a *semantic* property). So, re-using D-sequents is not as easy as re-using conflict clauses in SAT-solving. In this paper, we address this problem. We introduce a new definition of D-sequents that enables their re-usability. We develop a theory showing under what conditions a D-sequent can be safely re-used.

#### 1 Introduction

Many verification problems can be cast as an instance of the Quantifier Elimination (QE) problem or its variations<sup>1</sup>. So any progress in solving the QE problem is of great importance. In this paper, we consider the QE problem for propositional CNF formulas with existential quantifiers. Given formula  $\exists X[F(X,Y)]$  where X and Y are sets of variables, the QE problem is to find a quantifier-free formula  $F^*(Y)$  such that  $F^* \equiv \exists X[F]$ . In [3,4], we introduced a new approach to QE based on the following observation. Let us call a clause<sup>2</sup> of F an X-clause if it contains at least one variable of X. Solving the QE problem  $\exists X[F(X,Y)]$  reduces to finding formula  $F^*(Y)$  implied by F that makes the X-clauses of F redundant in  $\exists X[F^* \land F]$  (and so  $F^* \equiv \exists X[F]$  holds).

<sup>&</sup>lt;sup>1</sup> In [5], we introduced Partial QE (PQE) where only a *part* of the formula is taken out of the scope of quantifiers. The appeal of PQE is twofold. First, many verification problems like equivalence and model checking require partial rather than complete QE [1,2]. Second, PQE is much simpler to solve than QE. Both QE and PQE benefit from the results of this paper. However, since QE is conceptually simpler than PQE, we picked the former to introduce our new approach to D-sequent re-using.

<sup>&</sup>lt;sup>2</sup> A clause is a disjunction of literals. So, a CNF formula is a conjunction of clauses.

To implement the approach above, we introduced an algorithm called DCDS (Derivation of Clause D-Sequents). DCDS is based on the following three ideas. First, DCDS branches on variables of F to reach a subspace where proving redundancy<sup>3</sup> of X-clauses (or making them redundant by adding a new clause) is easy. Second, once an X-clause is proved redundant, DCDS stores this fact in the form of a Dependency Sequent (**D-sequent**). A D-sequent is a record  $(\exists X[F], \vec{q}) \to C$  where C is an X-clause of F and  $\vec{q}$  is an assignment to variables of F. This record states that C is redundant in  $\exists X[F]$  in subspace  $\vec{q}$ . The third idea of DCDS is to use a resolution-like operation called join to merge the results of branches. This join operation is applied to D-sequents  $(\exists X[F], \vec{q'}) \to C$  and  $(\exists X[F], \vec{q''}) \to C$  derived in branches v = 0 and v = 1 where v is a variable of F. The result of this operation is a D-sequent  $(\exists X[F], \vec{q}) \to C$  where  $\vec{q}$  does not contain variable v.

To make DCDS more efficient, it is natural to try to re-use a D-sequent  $(\exists X[F], \vec{q}) \to C$  in every subspace  $\vec{r}$  where  $\vec{q} \subseteq \vec{r}$  (i.e.  $\vec{r}$  contains all the assignments of  $\vec{q}$ ). However, here one faces the following problem. The definition of D-sequent  $(\exists X[F], \vec{q}) \to C$  implies that C is also redundant in subspace  $\vec{q}$  for formulas  $\exists X[G]$  logically equvialent to  $\exists X[F]$  where G is a subset of F. However, this may not be true for some formulas  $\exists X[G]$ . Here is a simple example of that. Let formula  $\exists X[F]$  contain two identical X-clauses C' and C''. Then D-sequents  $(\exists X[F], \emptyset) \to C'$  and  $(\exists X[F], \emptyset) \to C''$  hold. They state that C' and C'' are redundant in  $\exists X[F]$  individually. However, in general, one cannot A drop both A and A from A

The problem above prevents DCDS from reusing D-sequents. The reason why D-sequents cannot be re-used as easily as, say, conflict clauses in SAT-solvers [8,9] is as follows. Redundancy of a clause in a formula is a **structural** property<sup>5</sup>. That is the fact that clause C is redundant in formula F may not hold in a formula F' logically equivalent to F. On the other hand, re-using a conflict clause C is based on the fact that C is implied by the initial formula F and implication is a **semantic** property. That is C is implied by every formula F' logically equivalent to F.

In this paper, we address the problem of re-usability of D-sequents. Our approach is based on the following observation. Consider the example above with two identical X-clauses C', C''. The D-sequent  $(\exists X[F], \emptyset) \to C'$  requires the presence of clause  $C'' \in F$ . This means that C'' is supposed to be proved redundant after C'. On the contrary, the D-sequent  $(\exists X[F], \emptyset) \to C''$  requires

<sup>&</sup>lt;sup>3</sup> An X-clause C is said to be redundant in  $\exists X[F]$  if  $\exists X[F] \equiv \exists X[F \setminus \{C\}]$ . In this paper, we use the standard convention of viewing a set of clauses  $\{C_1, \ldots, C_n\}$  as an alternative way to specify the CNF formula  $C_1 \wedge \cdots \wedge C_n$ . So, the expression  $H \setminus \{C\}$  denotes the CNF formula obtained from H by removing clause C.

<sup>&</sup>lt;sup>4</sup> That is  $\exists X[F] \not\equiv \exists X[F \setminus \{C', C''\}].$ 

<sup>&</sup>lt;sup>5</sup> This is true regardless of whether this formula has quantifiers.

the presence of  $C' \in F$  and hence C'' is proved redundant before C'. So these D-sequents have a conflict in the order of proving redundancy of C' and C''.

To be able to identify order conflicts, we modify the definition of D-sequents given in [4]. A new D-sequent S is a record  $(\exists X[F], \vec{q}, H) \to C$  where H is a subset of  $F \setminus \{C\}$ . This D-sequent states that the clause C is redundant in subspace  $\vec{q}$  in every formula  $\exists X[W]$  where  $\exists X[W] \equiv \exists X[F]$  in subspace  $\vec{q}$  and  $(H \cup \{C\}) \subseteq W \subseteq F$ . Note that if an X-clause of H is proved redundant and removed from F, the D-sequent S is not applicable. So one can view H as an **order constraint** stating that S applies only if X-clauses of H are proved redundant after C. In other words, one can safely reuse S in subspace  $\vec{r}$  where  $\vec{q} \subseteq \vec{r}$ , if none of the X-clauses of H is proved redundant yet.

The contribution of this paper is as follows. First, we give the necessary definitions and propositions explaining the semantics of D-sequents stating redundancy of X-clauses<sup>6</sup> (Sections 2, 3, 4 and 5.) Second, we give a new definition of D-sequents facilitating their re-using (Section 6). We also introduce the notion of consistent D-sequents and show that they can be re-used. Third, we re-visit definitions of atomic D-sequents (i.e. D-sequents stating trivial cases of redundancy) and the join operation to accommodate the D-sequents of the new kind (Sections 7 and 8). Fourth, we present  $DCDS^+$ , a version of DCDS that can safely re-use D-sequents (Section 12).

#### 2 Basic Definitions

In this paper, we consider only propositional CNF formulas. In the sequel, when we say "formula" without mentioning quantifiers we mean a quantifier-free CNF formula. We consider true and false as a special kind of clauses. A non-empty clause C becomes true when it is satisfied by an assignment  $\vec{q}$  i.e. when a literal of C is set to true by  $\vec{q}$ . A clause C becomes false when it is falsified by  $\vec{q}$  i.e. when all the literals of C are set to false by  $\vec{q}$ .

**Definition 1.** Let F be a CNF formula and X be a subset of variables of F. We will refer to formula  $\exists X[F]$  as  $\exists \mathbf{CNF}$ .

**Definition 2.** Let  $\vec{q}$  be an assignment and F be a CNF formula.  $Vars(\vec{q})$  denotes the variables assigned in  $\vec{q}$ ; Vars(F) denotes the set of variables of F;  $Vars(\exists X[F])$  denotes  $Vars(F) \setminus X$ .

**Definition 3.** Let C be a clause, H be a formula that may have quantifiers, and  $\vec{p}$  be an assignment.  $C_{\vec{p}}$  is true if C is satisfied by  $\vec{p}$ ; otherwise it is the clause obtained from C by removing all literals falsified by  $\vec{p}$ .  $H_{\vec{p}}$  denotes the formula obtained from H by replacing C with  $C_{\vec{p}}$ .

<sup>&</sup>lt;sup>6</sup> In [4], we just give basic definitions. In [6], we do provide a detailed theoretical consideration but it is meant for D-sequents introduced in [3] expressing redundancy of variables rather than clauses.

**Definition 4.** Let G, H be formulas that may have quantifiers. We say that G, H are **equivalent**, written  $G \equiv H$ , if for all assignments  $\vec{q}$  such that  $Vars(\vec{q}) \supseteq (Vars(G) \cup Vars(H))$ , we have  $G_{\vec{q}} = H_{\vec{q}}$ .

**Definition 5.** The Quantifier Elimination (QE) problem for  $\exists CNF$  formula  $\exists X[F(X,Y)]$  is to find a formula  $F^*(Y)$  such that  $F^* \equiv \exists X[F]$ .

Remark 1. From now on, we will use Y and X to denote sets of free and quantified variables respectively. We will assume that variables denoted by  $x_i$  and  $y_i$  are in X and Y respectively. When we use Y and X in a quantifier-free formula we mean that, in the context of QE, the set X specifies the quantified variables.

**Definition 6.** A clause C of F is called a **Z-clause** if  $Vars(C) \cap Z \neq \emptyset$ . Denote by  $\mathbf{F}^{\mathbf{Z}}$  the set of all Z-clauses of F.

**Definition 7.** Let F be a CNF formula and  $G \subseteq F$  (i.e. G is a non-empty subset of clauses of F). The clauses of G are **redundant** in F if  $F \equiv (F \setminus G)$ . The clauses of G are **redundant** in formula  $\exists X[F]$  if  $\exists X[F] \equiv \exists X[F \setminus G]$ .

Note that  $F \equiv (F \setminus G)$  implies  $\exists X[F] \equiv \exists X[F \setminus G]$  but the opposite is not true.

## 3 Clause Redundancy And Boundary Points

In this section, we explain the semantics of QE in terms so-called boundary points.

**Definition 8.** Given assignment  $\vec{p}$  and a formula F, we say that  $\vec{p}$  is a **point** of F if  $Vars(F) \subseteq Vars(\vec{p})$ .

In the sequel, by "assignment" we mean a possibly partial one. To refer to a full assignment we will use the term "point".

**Definition 9.** Let F be a formula and  $Z \subseteq Vars(F)$ . A point  $\vec{p}$  of F is called a **Z-boundary point** of F if a)  $Z \neq \emptyset$  and b)  $F_{\vec{p}} = false$  and c) every clause of F falsified by  $\vec{p}$  is a Z-clause and d) the previous condition breaks for every proper subset of Z.

Remark 2. Let F(X,Y) be a CNF formula where sets X and Y are interpreted as described in Remark 1. In the context of QE, we will deal exclusively with Z-boundary points that falsify only X-clauses of F and so  $Z \subseteq X$  holds.

Example 1. Let  $X = \{x_1, x_2, x_3\}$  and  $Y = \{y_1, y_2\}$ . Let F(X, Y) be a CNF formula of four clauses:  $C_1 = x_1 \lor x_2$ ,  $C_2 = \overline{x_1} \lor y_1$ ,  $C_3 = x_1 \lor \overline{x_3} \lor y_2$ ,  $C_4 = \overline{x_2} \lor y_2$ . The clauses of F falsified by  $\vec{p} = (x_1 = 0, x_2 = 0, x_3 = 1, y_1 = 0, y_2 = 0)$  are  $C_1$  and  $C_3$ . One can verify that  $\vec{p}$  and the set  $Z = \{x_1\}$  satisfy the four conditions of Definition 9, which makes  $\vec{p}$  a  $\{x_1\}$ -boundary point. The set Z above is not unique. One can easily check that  $\vec{p}$  is also a  $\{x_2, x_3\}$ -boundary point.

The term "boundary" is justified as follows. Let F be a satisfiable CNF formula with at least one clause. Then there always exists a  $\{v\}$ -boundary point of F,  $v \in Vars(F)$  that is different from a satisfying assignment only in value of v.

**Definition 10.** Given a CNF formula F(X,Y) and a Z-boundary point  $\vec{p}$  of F:

- $\vec{p}$  is X'-removable in F if 1)  $Z \subseteq X' \subseteq X$ ; and 2) there is a clause C such that a)  $F \Rightarrow C$ ; b)  $Vars(C) \cap X' = \emptyset$ ; and c)  $C_{\vec{p}} = false$ .
- $\vec{p}$  is **removable** in  $\exists X[F]$  if  $\vec{p}$  is X-removable in F.

In the above definition, notice that  $\vec{p}$  is not a Z-boundary point of  $F \wedge C$  because  $\vec{p}$  falsifies C and  $Vars(C) \cap Z = \emptyset$ . So adding clause C to F eliminates  $\vec{p}$  as a Z-boundary point.

Example 2. Let us consider the  $\{x_1\}$ -boundary point  $\vec{p}=(x_1=0,x_2=0,x_3=1,y_1=0,y_2=0)$  of Example 1. Let C denote clause  $C=y_1\vee y_2$  obtained by resolving  $C_1$ ,  $C_2$  and  $C_4$  on variables  $x_1$  and  $x_2$ . Note that set  $Z=\{x_1\}$  and C satisfy the conditions a),b) and c) of Definition 10 for X'=X. So  $\vec{p}$  is an X-removable  $\{x_1\}$ -boundary point. After adding C to F,  $\vec{p}$  is not an  $\{x_1\}$ -boundary point any more. Let us consider the point  $\vec{q}=(x_1=0,x_2=0,x_3=1,y_1=1,y_2=1)$  obtained from  $\vec{p}$  by flipping values of  $y_1$  and  $y_2$ . Both  $\vec{p}$  and  $\vec{q}$  have the same set of falsified clauses consisting of  $C_1$  and  $C_3$ . So, like  $\vec{p}$ , point  $\vec{q}$  is an  $\{x_1\}$ -boundary point. However, no clause C implied by F and consisting only of variables of Y is falsified by  $\vec{q}$ . So, the latter, is an  $\{x_1\}$ -boundary point that is not X-removable.

**Proposition 1.** A Z-boundary point  $\vec{p}$  of F(X,Y) is removable in  $\exists X[F]$ , iff one cannot turn  $\vec{p}$  into an assignment satisfying F by changing only the values of variables of X.

The proofs are given in the appendix.

**Proposition 2.** Let F(X,Y) be a CNF formula where  $F^X \neq \emptyset$  (see Definition 6). Let G be a non-empty subset of  $F^X$ . The set G is not redundant in  $\exists X[F]$  iff there is a Z-boundary point  $\vec{p}$  of F such that a) every clause falsified by  $\vec{p}$  is in G and b)  $\vec{p}$  is X-removable in F.

Proposition 2 justifies the following strategy of solving the QE problem. Add to F a set G of clauses that a) are implied by F; b) eliminate every X-removable boundary point falsifying a subset of X-clauses of F. By dropping all X-clauses of F, one produces a solution to the QE problem.

#### 4 Quantifier Elimination By Branching

In this section, we explain the semantics of QE algorithm called DCDS [4] (Derivation of Clause D-Sequents). A high-level description of DCDS is given

<sup>&</sup>lt;sup>7</sup> See Definition 13 of the resolution operation.

in Section 11. DCDS is a branching algorithm. Given a formula  $\exists X[F]$ , DCDS branches on variables of F until it proves that every X-clause is redundant in the current subspace. (In case of a conflict, proving X-clauses of F redundant, in general, requires adding to F a conflict clause.) Then DCDS merges the results obtained in different branches to prove that the X-clauses are redundant in the entire search space. Below we give propositions justifying the strategy of DCDS. Proposition 3 shows how to perform elimination of removable boundary points of F in the subspace specified by assignment  $\vec{q}$ . This is done by using formula  $F_{\vec{q}}$ , a "local version" of F. Proposition 4 justifies proving redundancy of X-clauses of  $F_{\vec{q}}$  incrementally.

Let  $\vec{q}$  and  $\vec{r}$  be assignments to a set of variables Z. Since  $\vec{q}$  and  $\vec{r}$  are sets of value assignments to individual variables of Z one can apply set operations to them. We will denote by  $\vec{r} \subseteq \vec{q}$  the fact that  $\vec{q}$  contains all value assignments of  $\vec{r}$ . The assignment consisting of value assignments of  $\vec{q}$  and  $\vec{r}$  is represented as  $\vec{q} \cup \vec{r}$ .

**Proposition 3.** Let  $\exists X[F(X,Y)]$  be an  $\exists \text{CNF}$  and  $\vec{q}$  be an assignment to Vars(F). Let  $\vec{p}$  be a Z-boundary point of F where  $\vec{q} \subseteq \vec{p}$  and  $Z \subseteq X$ . Then if  $\vec{p}$  is removable in  $\exists X[F]$  it is also removable in  $\exists X[F_{\vec{q}}]$ .

Remark 3. One cannot reverse Proposition 3: a boundary point may be X-removable in  $F_{\vec{q}}$  and not X-removable in F. For instance, if X = Vars(F), a Z-boundary point  $\vec{p}$  of F where |Z| = 1 is removed from  $\exists X[F]$  only by adding an empty clause to F. So if F is satisfiable,  $\vec{p}$  is not removable. Yet  $\vec{p}$  may be removable in  $\exists X[F_{\vec{q}}]$  if  $F_{\vec{q}}$  is unsatisfiable. A ramification of the fact that Proposition 3 is not reversible is discussed in Section 5.

**Proposition 4.** Let  $\exists X[F(X,Y)]$  be an  $\exists CNF$  and  $H \subset F^X$  be redundant in  $\exists X[F]$ . Let an X-clause C of  $F \setminus H$  be redundant in  $\exists X[F \setminus H]$ . Then  $H \cup \{C\}$  is redundant in  $\exists X[F]$ .

Remark 4. To simplify the notation, we will sometimes use the expression "clause C is redundant in  $\exists X[F]$  in subspace  $\vec{q}$ " instead of saying "clause  $C_{\vec{q}}$  is redundant in  $\exists X[F_{\vec{q}}]$ ".

Proposition 4 shows that one can prove redundancy of, say, a set of X-clauses  $\{C', C''\}$  in  $\exists X[F]$  in subspace  $\vec{q}$  incrementally. This can be done by a) proving redundancy of C' in  $\exists X[F]$  in subspace  $\vec{q}$  and c) proving redundancy of C'' in formula  $\exists X[F \setminus \{C'\}]$  in subspace  $\vec{q}$ .

#### 5 Virtual redundancy

If a boundary point  $\vec{p}$  is X-removable in  $\exists X[F_{\vec{q}}]$ , this does not mean that it is X-removable in  $\exists X[F]$  (see Remark 3). This fact leads to the following problem. Let  $\vec{q}$  and  $\vec{r}$  be two assignments to Vars(F) and  $\vec{q} \subset \vec{r}$ . Suppose that clause C is redundant in  $\exists X[F]$  in subspace  $\vec{q}$ . It is natural to expect that this also holds in the smaller subspace  $\vec{r}$ . However,  $\exists X[F_{\vec{q}}] \equiv \exists X[F_{\vec{q}} \setminus C_{\vec{q}}]$  does not imply

 $\exists X[F_{\vec{r}}] \equiv \exists X[F_{\vec{r}} \setminus C_{\vec{r}}]$ . In particular, due to this problem, one cannot define the join operation in terms of redundancy specified by Definition 7. To address this issue we introduce the notion of *virtual* redundancy.

**Definition 11.** Let  $\exists X[F(X,Y)]$  be an  $\exists CNF$  formula,  $\vec{q}$  be an assignment to Vars(F), and  $C_{\vec{q}}$  be an X-clause of  $F_{\vec{q}}$ . Let B be the set of points of F such every  $\vec{p} \in B$  falsifies only clause C and is X-removable. Clause  $C_{\vec{q}}$  is called **virtually redundant** in  $\exists X[F_{\vec{q}}]$  if one of the two conditions are true.

- 1.  $B = \emptyset$  or
- 2. For every  $\vec{p} \in B$ , there is an assignment  $\vec{r}$  where  $\vec{q^*} \subseteq \vec{r} \subset \vec{q}$  such that  $\vec{p}$  is not X-removable in  $F_{\vec{r}}$ . Here  $\vec{q^*}$  is obtained from  $\vec{q}$  by removing all value assignments to variables of X.

The first condition just means that  $\exists X[F_{\vec{q}} \setminus C_{\vec{q}}] \equiv \exists X[F_{\vec{q}}]$ . We will refer to this type of redundancy (earlier specified by Definition 7) as regular redundancy. Regular redundancy is a special case of virtual redundancy.

**Proposition 5.** Let  $\vec{q}$  be an assignment to Vars(F) and clause  $C_{\vec{q}}$  be redundant in  $\exists X[F_{\vec{q}}]$ . Then, for every  $\vec{r}$  such that  $\vec{q} \subset \vec{r}$ , clause  $C_{\vec{r}}$  is virtually redundant in  $\exists X[F_{\vec{r}}]$ .

From now on, when we say that a clause  $C_{\vec{r}}$  is redundant in  $\exists X[F_{\vec{r}}]$  we mean that it is at least *virtually redundant*. Note that, in general, proving virtual redundancy of C in subspace  $\vec{r}$  can be extremely hard. We avoid this problem by using the notion of virtual of redundancy only if we have *already proved* that C is redundant in a subspace containing subspace  $\vec{r}$ . (For instance, we have already proved that C is redundant in  $\exists X[F]$  in subspace  $\vec{q}$  where  $\vec{q} \subset \vec{r}$ .)

#### 6 Dependency Sequents (D-sequents)

In this section, we give a new definition of D-sequents that is different from that of [4].

**Definition 12.** Let  $\exists X[F]$  be an  $\exists CNF$  formula. Let  $\vec{q}$  be an assignment to Vars(F) and  $C \in F^X$  and  $H \subseteq (F \setminus \{C\})$ . A dependency sequent (**D-sequent**) S has the form  $(\exists X[F], \vec{q}, H) \to C$ . It states that clause  $C_{\vec{q}}$  is redundant in every formula  $\exists X[W_{\vec{q}}]$  logically equivalent to  $\exists X[F_{\vec{q}}]$  where  $H \cup \{C\} \subseteq W \subseteq F$ . The assignment  $\vec{q}$  and formula H are called the **conditional** and **order constraint** of S respectively. We will refer to W as a **member formula** for S.

Definition 12 implies that the D-sequent S becomes inapplicable if a clause of H is removed from F. So, S is meant to be used in situations where the X-clauses of H are proved redundant after C (hence the name "order constraint"). As we mentioned in the introduction, in [4], a D-sequent implies redundancy of clause C in  $\exists X[F]$  and in (some) logically equivalent formulas  $\exists X[W]$  where  $W \subseteq F$ . In Definition 12, the set of formulas  $\exists X[W]$  where C is redundant in subspace  $\vec{q}$ 

is specified *precisely*. We will say that a D-sequent  $(\exists X[F], \vec{q}, H) \to C$  is **fragile** if H contains at least one X-clause. Such a D-sequent becomes inapplicable if an X-clause of H is proved redundant before C. If H does not contain X-clauses, the D-sequent above is called **robust**. A robust D-sequent is not affected by the order in which X-clauses are proved redundant.

Remark 5. We will abbreviate D-sequent  $(\exists X[F], \vec{q}, H) \to C$  to  $(\vec{q}, H) \to C$  if formula  $\exists X[F]$  is known from the context. We will further reduce  $(\vec{q}, H) \to C$  to  $\vec{q} \to C$  if  $H = \emptyset$  i.e. if no order constraint is imposed.

There are two ways to produce D-sequents. First, one can generate an "atomic" D-sequent that states a trivial case of redundancy. The three atomic types of D-sequents are presented in Section 7. Second, one can use a pair of existing D-sequents to generate a new one by applying a resolution-like operation called *join* (Section 8).

### 7 Atomic D-sequents

In this section we describe D-sequents called atomic. These D-sequents are generated when redundancy of a clause can be trivially proved. Similarly to [4], we introduce atomic D-sequents of three kinds. However, in contrast to [4], we consider D-sequents specified by Definition 12. In particular, we show that D-sequents of the first kind are robust whereas D-sequents of the second and third kind are fragile.

#### 7.1 Atomic D-sequents of the first kind

**Proposition 6.** Let  $\exists X[F]$  be an  $\exists CNF$  and  $C \in F$  and  $v \in Vars(C)$ . Let assignment v = b where  $b \in \{0,1\}$  satisfy C. Then D-sequent  $(v = b) \rightarrow C$  holds. We will refer to it as an atomic D-sequent of the **first kind**.

Example 3. Let  $\exists X[F]$  be an  $\exists CNF$  and  $C = \overline{x}_1 \lor y_5$  be a clause of F. Since C is satisfied by assignments  $x_1 = 0$  and  $y_5 = 1$ , D-sequents  $(x_1 = 0) \to C$  and  $(y_5 = 1) \to C$  hold.

#### 7.2 Atomic D-sequents of the second kind

**Proposition 7.** Let  $\exists X[F]$  be an  $\exists \text{CNF}$  formula and  $\vec{q}$  be an assignment to Vars(F). Let B, C be two clauses of F. Let  $C_{\vec{q}}$  be an X-clause and  $B_{\vec{q}}$  imply  $C_{\vec{q}}$  (i.e. every literal of  $B_{\vec{q}}$  is in  $C_{\vec{q}}$ ). Then the D-sequent  $(\vec{q}, H) \to C$  holds where  $H = \{B\}$ . We will refer to it as an atomic D-sequent of the **second kind**.

Example 4. Let  $\exists X[F]$  be an  $\exists CNF$  formula. Let  $B = y_1 \vee x_2$  and  $C = x_2 \vee \overline{x_3}$  be X-clauses of F. Let  $\vec{q} = (y_1 = 0)$ . Since  $B_{\vec{q}}$  implies  $C_{\vec{q}}$ , the D-sequent  $(\vec{q}, \{B\}) \to C$  holds. Since  $B_{\vec{q}}$  is an X-clause, this D-sequent is fragile.

#### 7.3 Atomic D-sequents of the third kind

To introduce atomic D-sequents of the third kind, we need to make a few definitions.

**Definition 13.** Let C' and C'' be clauses having opposite literals of exactly one variable  $v \in Vars(C') \cap Vars(C'')$ . The clause C consisting of all literals of C' and C'' but those of v is called the **resolvent** of C', C'' on v. Clause C is said to be obtained by **resolution** on v. Clauses C', C'' are called **resolvable** on v.

**Definition 14.** A clause C of a CNF formula F is called **blocked** at variable v, if no clause of F is resolvable with C on v. The notion of blocked clauses was introduced in [7].

**Proposition 8.** Let  $\exists X[F]$  be an  $\exists CNF$  formula. Let C be an X-clause of F and  $v \in Vars(C) \cap X$ . Let  $C_1, \ldots, C_k$  be the clauses of F that can be resolved with C on variable v. Let  $(\vec{q}_1, H_1) \to C_1, \ldots, (\vec{q}_k, H_k) \to C_k$  be a consistent set i = k of i = k of i = k. D-sequents. Then i = k of i = k and i = k if i =

Note that, in general, a D-sequent of the third kind is fragile.

Example 5. Let  $\exists X[F(X,Y)]$  be an  $\exists \text{CNF}$  formula. Let  $C_3, C_6, C_8$  be the only clauses of F with variable  $x_5 \in X$  where  $C_3 = x_5 \land x_{10}, C_6 = \overline{x}_5 \land y_1, C_8 = \overline{x_5} \lor y_3 \lor y_5$ . Note that assignment  $y_1 = 1$  satisfies clause  $C_6$ . So the D-sequent  $(y_1 = 1) \to C_6$  holds. Suppose that D-sequent  $(\vec{r}, \{C_{10}\}) \to C_8$  holds where  $C_{10}$  is a clause of F and  $\vec{r} = (y_2 = 0, x_{10} = 1)$ . From Proposition 8 it follows that D-sequent  $(\vec{q}, \{C_{10}\}) \to C_3$  holds where  $\vec{q} = (y_1 = 1, y_2 = 0, x_{10} = 1)$ .

#### 8 Join Operation

In this section, we describe the operation of joining D-sequents that produces a new D-sequent from two parent D-sequents. In contrast to [4], the join operation introduced here is applied to D-sequents with order constraints.

**Definition 15.** Let  $\vec{q'}$  and  $\vec{q''}$  be assignments in which exactly one variable  $v \in Vars(\vec{q'}) \cap Vars(\vec{q''})$  is assigned different values. The assignment  $\vec{q}$  consisting of all the value assignments of  $\vec{q'}$  and  $\vec{q''}$  but those to v is called the **resolvent** of  $\vec{q'}$ ,  $\vec{q''}$  on v. Assignments  $\vec{q'}$ ,  $\vec{q''}$  are called **resolvable** on v.

**Proposition 9.** Let  $\exists X[F]$  be an  $\exists CNF$  formula for which D-sequents  $(\vec{q'}, H') \rightarrow C$  and  $(\vec{q''}, H'') \rightarrow C$  hold. Let  $\vec{q'}, \vec{q''}$  be resolvable on  $v \in Vars(F)$  and  $\vec{q}$  be the resolvent of  $\vec{q'}$  and  $\vec{q''}$ . Let  $H = H' \cup H''$ . Then the D-sequent  $(\vec{q}, H) \rightarrow C$  holds.

<sup>&</sup>lt;sup>8</sup> We will introduce the notion of a consistent set of D-sequents later, see Definition 19. Consistency of D-sequents in Proposition 8 means that  $C_1, \ldots, C_k$  are redundant together in subspace  $\vec{q} = \bigcup_{i=1}^{i=k} \vec{q}_i$ . So clause C is blocked at variable v in subspace  $\vec{q}$ .

**Definition 16.** We will say that the D-sequent  $(\vec{q}, H) \to C$  of Proposition 9 is produced by **joining D-sequents**  $(\vec{q'}, H') \to C$  and  $(\vec{q''}, H'') \to C$  at v.

Remark 6. Note that the D-sequent S produced by the join operation has a stronger order constraint than its parent D-sequents. The latter have order constraints H' and H'' in subspaces v=0 and v=1, whereas S has the same order constraint  $H=H'\cup H''$  in either subspace. Due to this "imprecision" of the join operation, a set of D-sequents with conflicting order constraints can still be correct (see Section 9 and Subsection 11.2).

## 9 Re-usability of D-sequents

To address the problem of D-sequent re-using, we introduce the notion of composability. Informally, a set of D-sequents is composable if the clauses stated redundant individually are also redundant *collectively*. Robust D-sequents are always composable. So they can be re-used in any context like conflict clauses in SAT-solvers. However, this is not true for fragile D-sequents. Below, we show that such D-sequents are composable if they are consistent. So it is safe to re-use a fragile D-sequent in a subspace  $\vec{q}$ , if it is consistent with the D-sequents already used in subspace  $\vec{q}$ .

**Definition 17.** Assignments  $\vec{q'}$  and  $\vec{q''}$  are called **compatible** if every variable from  $Vars(\vec{q'}) \cap Vars(\vec{q''})$  is assigned the same value.

**Definition 18.** Let  $\exists X[F]$  be an  $\exists CNF$ . A set of D-sequents  $(\vec{q}_1, H_1) \rightarrow C_1, \ldots, (\vec{q}_k, H_k) \rightarrow C_k$  is called **composable** if the clauses  $\{C_1, \ldots, C_k\}$  are redundant collectively as well. That is  $\exists X[F] \equiv \exists X[F \setminus \{C_1, \ldots, C_k\}]$  holds in subspace  $\vec{q}$  where  $\vec{q} = \bigcup_{i=1}^{i=k} \vec{q}_i$ .

**Definition 19.** Let  $\exists X[F]$  be an  $\exists CNF$ . A set of D-sequents  $(\vec{q}_1, H_1) \rightarrow C_1, \ldots, (\vec{q}_k, H_k) \rightarrow C_k$  is called **consistent** if

- every pair of assignments  $\vec{q_i}, \vec{q_j}, 1 \leq i, j \leq k$  is compatible;
- there is a total order  $\pi$  over clauses of  $\bigcup_{i=1}^{i=k} H_i \cup \{C_i\}$  that satisfies the order constraints of these D-sequents i.e.  $\forall C \in H_i, \ \pi(C_i) < \pi(C)$  holds where  $i = 1, \ldots, k$ .

**Proposition 10.** Let  $\exists X[F]$  be an  $\exists CNF$ . Let  $(\vec{q}_1, H_1) \to C_1, \ldots, (\vec{q}_k, H_k) \to C_k$  be a consistent set of D-sequents. Then these D-sequents are composable and hence clauses  $\{C_1, \ldots, C_k\}$  are collectively redundant in  $\exists X[F]$  in subspace  $\vec{q}$  where  $\vec{q} = \bigcup_{i=1}^{i=k} \vec{q}_i$ .

Remark 7. The fact that D-sequents  $S_1, \ldots, S_k$  are inconsistent does not necessarily mean that these D-sequents are not composable. As we mentioned in Remark 6, as far as order constraints are concerned, the join operation is not "precise". This means that if the D-sequents above are obtained by applying the join operation, their order-inconsistency may be artificial. An example of that is the QE procedure called DCDS [4]. As we explain in Subsection 11.2, if one uses the new definition of D-sequents (i.e. Definition 12), the D-sequents produced by DCDS are, in general, inconsistent. However, DCDS is provably correct [6].

Remark 8. Let  $\exists X[F(X,Y)]$  be an  $\exists \text{CNF}$  and R(X,Y) be the set of clauses added to F by a QE-solver. Let  $F^X \cup R^X = \{C_1, \ldots, C_k\}$  (i.e. the latter is the set of all X-clauses of  $F \cup R$ ). This QE-solver terminates when the set  $R \setminus R^X$  is sufficient to derive consistent D-sequents  $(\exists X[F \land R],\emptyset,H_1) \to C_1,\ldots,(\exists X[F \land R],\emptyset,H_k) \to C_k$ . From Proposition 10 it follows, that all X-clauses can be dropped from  $\exists X[F \land R]$ . The resulting formula  $F^*(Y)$  consisting of clauses of  $F \cup R \setminus \{C_1,\ldots,C_k\}$  is logically equivalent to  $\exists X[F]$ .

#### 10 Two Useful Transformations Of D-sequents

In this section, we describe two transformations that are useful for a QE-solver based on the machinery of D-sequents. Since a QE-solver has to add new clauses once in a while, D-sequents of different branches are, in general, computed with respect to different formulas. In Subsection 10.1, we describe a transformation meant for "aligning" such D-sequents. In Subsection 10.2, we describe a transformation meant for relaxing the order constraint of a D-sequent. In Section 12, this transformation is used to generate a consistent set of D-sequents.

#### 10.1 D-sequent alignment

According to Definition 12, a D-sequent holds with respect to a particular  $\exists \text{CNF}$  formula  $\exists X[F]$ . Proposition 11 shows that this D-sequent also holds after adding to F implied clauses.

**Proposition 11.** Let D-sequent  $(\exists X[F], \vec{q}, H) \to C$  hold and R be a CNF formula implied by F. Then D-sequent  $(\exists X[F \land R], \vec{q}, H) \to C$  holds too.

Proposition 11 is useful in aligning D-sequents derived in different branches. Suppose that  $(\exists X[F], \vec{q'}, H') \to C$  is derived in the current branch of the search tree where the last assignment is v=0. Suppose that  $(\exists X[F \land R], \vec{q''}, H'') \to C$  is derived after flipping the value of v from 0 to 1. Here R is the set of clauses implied by F that has been added to F before the second D-sequent was derived. One cannot apply the join operation to these D-sequents because they are computed with respect to different formulas. Proposition 11 allows one to replace  $(\exists X[F], \vec{q'}, H') \to C$  with  $(\exists X[F \land R], \vec{q'}, H') \to C$ . The latter can be joined with  $(\exists X[F \land R], \vec{q''}, H'') \to C$  at variable v.

<sup>&</sup>lt;sup>9</sup> This set consists of the clauses of R that depend only on variables of Y.

#### 10.2 Making a D-sequent more robust

In this subsection, we give two propositions showing how one can make a D-sequent S more robust. Proposition 12 introduces a transformation that removes a clause from the order constraint of S possibly adding to the latter some other clauses. Proposition 13 describes a scenario where by repeatedly applying this transformation one can remove a clause from the order constraint of S without adding any other clauses.

**Proposition 12.** Let  $\exists X[F]$  be an  $\exists \text{CNF.}$  Let  $(\vec{q'}, H') \to C'$  and  $(\vec{q''}, H'') \to C''$  be two D-sequents forming a consistent set (see Definition 19). Let C'' be in H'. Then D-sequent  $(\vec{q}, H) \to C'$  holds where  $\vec{q} = \vec{q'} \cup \vec{q''}$  and  $H = (H' \setminus \{C'''\}) \cup H''$ .

**Proposition 13.** Let  $\exists X[F]$  be an  $\exists \text{CNF}$  and  $(\vec{q}_1, H_1) \to C_1, \ldots, (\vec{q}_k, H_k) \to C_k$  be consistent D-sequents where  $H_i \subseteq \{C_1, \ldots, C_k\}$ ,  $i = 1, \ldots, k$ . Assume, for the sake of simplicity, that the numbering order is consistent with the order constraints. Let  $C_m$  be in  $H_i$ . Then, by repeatedly applying the transformation of Proposition 12, one can produce D-sequent  $(\vec{q}, H_i \setminus \{C_m\}) \to C_i$  where  $\vec{q}_i \subseteq \vec{q} \subseteq \vec{q}_i \cup \bigcup_{j=m}^{j=k} \vec{q}_j$ .

## 11 Recalling DCDS

In [4], we described a QE algorithm called *DCDS* (Derivation of Clause D-Sequents) that did not re-use D-sequents. We Recall *DCDS* in Subsections 11.1 and 11.2.

#### 11.1 A brief description of DCDS

The pseudocode of DCDS is given<sup>10</sup> in Fig. 1. DCDS uses the old definition of a D-sequent lacking an order constraint. DCDS accepts three parameters: formula  $\exists X[F]$  (denoted as  $\Phi$ ), the current assignment  $\vec{q}$  and the set of active D-sequents  $\Omega$ . (If an X-clause of F is proved redundant in subspace  $\vec{q}$ , this fact is stated by a D-sequent. This D-sequent is called **active**). DCDS returns the final formula  $\exists X[F]$  (where F consists of the initial clauses and derived clauses implied by F) and the set  $\Omega$  of current active D-sequents.  $\Omega$  has an active D-sequent for every X-clause of F. The conditional of this D-sequent is a subset of  $\vec{q}$ . In the first call of DCDS, the initial formula  $\exists X[F]$  is used and  $\vec{q}$  and  $\Omega$  are empty sets. A solution  $F^*(Y)$  to the QE problem at hand is obtained by dropping the X-clauses of the final formula  $\exists X[F]$  and removing the quantifiers.

For the sake of simplicity, Figure 1 gives a very abstract view of DCDS. For instance, we omit the lines of code where new clauses are generated. Our objective here is just to show the part of DCDS where D-sequents are involved. A more detailed description of DCDS can be found in [4].

DCDS starts with examining the X-clauses whose redundancy is not proved yet. Namely, DCDS checks if the redundancy of such clauses can be established by atomic D-sequents (line 1) introduced in Section 7. If all X-clauses are proved redundant, DCDS terminates returning the current formula  $\exists X[F]$  and the current set of active D-sequents (lines 2-3). Otherwise, DCDS moves to the branching part of the algorithm (lines 4-9).

Fig. 1. DCDS procedure

First, DCDS picks a variable v to branch on (line 4). Then it explores the branch  $\vec{q} \cup \{v = 0\}$  (line 5). The set  $\Omega_0$  returned in this branch, in general, contains D-sequents whose conditionals include assignment v=0. These D-sequents are inapplicable in branch v = 1 and so they are discarded (line 6). After that DCDS explores branch  $\vec{q} \cup \{v = 1\}$  (line 7) returning a set of D-sequents  $\Omega_1$ . Then *DCDS* generates a set of D-sequents  $\Omega$  whose conditionals do not depend on v (line 8). Set  $\Omega$  consists of two parts. The first part comprises of the D-sequents of  $\Omega_0$  that do not depend on v. The second part consists of the D-sequents obtained by joining D-sequents of  $\Omega_0$  and  $\Omega_1$  that

do depend on v. Finally, DCDS terminates returning  $\exists X[F]$  and  $\Omega$ .

### 11.2 Correctness of DCDS

As we mentioned earlier, DCDS employs D-sequents introduced in [4] that lack order constraints. A D-sequent S of [4] states redundancy of a clause C in formula  $\exists X[F]$  in subspace  $\vec{q}$ . Besides, clause C is also assumed to be redundant in (some) formulas  $\exists X[W]$  logically equivalent to  $\exists X[F]$  where W is a subset of F. The problem here is that the set of formulas  $\exists X[W]$  for which S guranatees redundancy of clause C in subspace  $\vec{q}$  is not specified precisely. Nevertheless, DCDS is provably correct<sup>11</sup>.

There are three reasons why DCDS is correct despite the fact that it uses a "sloppy" definition of a D-sequent. First, DCDS does not re-use D-sequents. After generating a new D-sequent by the join operation, DCDS discards the parent clauses of this D-sequent. Second, in every branch of the search tree the X-clauses are proved redundant in some order which makes them composable. Third, by joining composable D-sequents obtained in branch v=0 with composable D-sequents obtained in branch v=1 one produces a set of composable D-sequents.

<sup>&</sup>lt;sup>11</sup> In [6], we proved the correctness of a similar algorithm. This proof applies to *DCDS*.

#### **12** Introducing $DCDS^+$

In this section, we describe a modification of *DCDS* that re-uses D-sequents. We will refer to it as  $DCDS^+$ . The pseudocode of  $DCDS^+$  is shown in Fig. 2. In comparison to DCDS,  $DCDS^+$  has one more input parameter: a set  $\Psi$  of D-sequents stored to re-use. The four lines where  $DCDS^+$  behaves differently from DCDS are marked with an asterisk.

```
// \Phi denotes \exists X[F]
// \vec{q} is an assignment to Vars(F)
// \Omega denotes the current set of active
         D-sequents
//\Psi denotes the set of stored D-sequents
DCDS^{+}(\Phi, \vec{q}, \Omega, \Psi){
      \Omega := \Omega \cup ReuseDseqs(\Phi, \vec{q}, \Psi, \Omega)
      \Omega := \Omega \cup AtomDseqs(\Phi, \vec{q}, \Omega)
      if (EveryXclauseRedund(\Phi, \Omega))
           \operatorname{return}(\Phi, \Omega, \Psi)
       _ _ _ _ _ _ _ _ _ _ .
      v := PickVar(F, \vec{q}, \Omega)
5
       (\Phi, \Omega_0, \Psi) := DCDS^+(\Phi, \vec{q} \cup (v = 0), \Omega, \Psi)
       \Omega := DropInapplic(\Omega_0, v)
       (\Phi, \Omega_1, \Psi) := DCDS^+(\Phi, \vec{q} \cup (v = 1), \Omega, \Psi)
       \Omega := JoinDseqs^+(\Phi, v, \Omega_0, \Omega_1)
10* \Psi := \Psi \cup DseqsToStore(\Omega)
      \operatorname{return}(\Phi, \Omega, \Psi)
```

Fig. 2.  $DCDS^+$  procedure

```
JoinDseqs^+(\Phi, v, \Omega_0, \Omega_1){
    \Omega := SymmDseqs(\Omega_0)
    G := FormXcls(\Omega_0 \setminus \Omega)
    for each (C \in G) {
3
         (S_0, S_1) := ExtrDseqs(\Omega_0, \Omega_1, C)
4
         S := join(S_0, S_1, v)
5
        if (incons(\Omega \cup \{S\}))
6
            S := FixDseq(S, \Omega, \Omega_0, \Omega_1)
7
         \Omega := \Omega \cup \{S\} \}
    return(\Omega);
```

Fig. 3. JoinDseqs<sup>+</sup> procedure

The difference between  $DCDS^+$ and *DCDS* is as follows. First,  $DCDS^+$  uses the new definition of D-sequents and thus keeps track of order constraints. (In particular,  $DCDS^+$  stores order constraints when generating the atomic Dsequents of the second/third kind. For that reason, line 2 is marked with an asterisk.) Second,  $DCDS^+$ tries to re-use D-sequents stored in  $\Psi$ . Namely, if an X-clause C is not proved redundant vet, DCDS+ checks if there is a D-sequent of  $\Psi$ a) that states redundancy of C; b) whose conditional  $\vec{r}$  satisfies  $\vec{r} \subseteq$  $\vec{q}$  and c) whose order constraint is consistent with those of active D-sequents. Third,  $DCDS^+$  stores some of new D-sequents obtained by the join operation (line 10).

The final difference between  $DCDS^+$ and DCDS is as follows. The order in which X-clauses are proved redundant in the two branches generated by splitting on variable v can be different. So if one just joins D-sequents obtained in those branches at variable v (as it is done by the JoinDseqs procedure of DCDS), an inconsistent set of D-sequents can be generated. If no D-sequents are re-used, as in DCDS, this inconsistency does not mean that the D-sequents of this set are not composable<sup>12</sup> (see Subsection 11.2). However, re-using D-sequents, as it is done in  $DCDS^+$ , may produce inconsistent

 $<sup>^{12}</sup>$  As we mentioned in Remark 6, an inconsistency can be introduced by the imprecision of the join operation with respect to order constraints.

D-sequents that are *indeed* not composable. For that reason, in  $DCDS^+$ , a modification of JoinDsegs called  $JoinDsegs^+$  is used.

The pseudocode of  $JoinDseqs^+$  is shown in Fig. 3. The objective of  $JoinDseqs^+$  is to generate a set of consistent D-sequents that do not depend on variable v. The resulting D-sequents are accumulated in  $\Omega$ .  $JoinDseqs^+$  starts by initializing  $\Omega$  with the D-sequents that are already symmetric in v i.e. their conditionals do not contain an assignment to variable v. Then  $JoinDseqs^+$  forms the set G of X-clauses whose D-sequents are asymmetric in v.

The main part of  $JoinDseqs^+$  consists of a loop (lines 3-8) where, for every clause C of G, a D-sequent whose conditional is symmetric in v is built. First,  $JoinDseqs^+$  extracts D-sequents  $S_0$  and  $S_1$  of clause C and joins them at variable v to produce a D-sequent S (lines 4-5). If the D-sequents of  $\Omega$  become inconsistent after adding S,  $JoinDseqs^+$  calls FixDseq to produce a D-sequent S that preserves the consistency of  $\Omega$ . Proposition 13 shows that it is always possible. Namely, one can always relax the order constraints of  $S_0$  and  $S_1$  thus relaxing that of S. In particular, one can totally eliminate order constraints of  $S_0$  and  $S_1$ , which makes them (and hence S) robust.

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## **Appendix**

The appendix contains proofs of the propositions listed in the paper. We also give proofs of lemmas used in the proofs of propositions.

## Propositions of Section 3: Clause Redundancy And Boundary Points

**Proposition 1.** A Z-boundary point  $\vec{p}$  of F(X,Y) is removable in  $\exists X[F]$ , iff one cannot turn  $\vec{p}$  into an assignment satisfying F by changing only the values of variables of X.

Proof: If part. Assume the contrary. That is  $\vec{p}$  is not removable while no satisfying assignment can be obtained from  $\vec{p}$  by changing only values of variables of X. Let C be the clause consisting of all variables of Y and falsified by  $\vec{p}$ . Since  $\vec{p}$  is not removable, clause C is not implied by F. This means that there is an assignment  $\vec{s}$  that falsifies C and satisfies F. By construction,  $\vec{s}$  and  $\vec{p}$  have identical assignments to variables of Y. Thus,  $\vec{s}$  can be obtained from  $\vec{p}$  by changing only values of variables of X and we have a contradiction.

Only if part. Assume the contrary. That is  $\vec{p}$  is removable but one can obtain an assignment  $\vec{s}$  satisfying F from  $\vec{p}$  by changing only values of variables of X. Since  $\vec{p}$  is removable, there is a clause C that is implied by F and falsified by  $\vec{p}$  and that depends only of variables of Y. Since  $\vec{s}$  and  $\vec{p}$  have identical assignments to variables of Y, point  $\vec{s}$  falsifies C. However, since  $\vec{s}$  satisfies F, this means that C is not implied by F and we have a contradiction.  $\Box$ 

**Proposition 2.** Let F(X,Y) be a CNF formula where  $F^X \neq \emptyset$  (see Definition 6). Let G be a non-empty subset of  $F^X$ . The set G is not redundant in  $\exists X[F]$  iff there is a Z-boundary point  $\vec{p}$  of F such that a) every clause falsified by  $\vec{p}$  is in G and b)  $\vec{p}$  is X-removable in F.

*Proof:* Let H denote  $F \setminus G$ . Given a point  $\vec{p}$ , let  $(\vec{x}, \vec{y})$  specify the assignments of  $\vec{p}$  to the variables of X and Y respectively.

If part. Assume the contrary, i.e., there is an X-removable point  $\vec{p} = (\vec{x}, \vec{y})$  of F but G is redundant in  $\exists X[F]$  and so  $\exists X[F] \equiv \exists X[H]$ . Since  $\vec{p}$  is a boundary point,  $F(\vec{p}) = 0$ . Since  $\vec{p}$  is removable,  $(\exists X[F])_{\vec{y}} = 0$ . On the other hand, since  $\vec{p}$  falsifies only clauses of G it satisfies H. Hence  $(\exists X[H])_{\vec{y}} = 1$  and  $(\exists X[F])_{\vec{y}} \neq (\exists X[H])_{\vec{y}}$ , which leads to a contradiction.

Only if part. Assume the contrary, i.e., set G is not redundant (and hence  $\exists X[F] \not\equiv \exists X[H]$ ) and there does not exist an X-removable Z-boundary point of F falsifying only clauses of G. Let  $\vec{y}$  be an assignment to Y such that  $(\exists X[F])_{\vec{y}} \neq (\exists X[H])_{\vec{y}}$ . Consider the following two cases.

- $(\exists X[F])_{\vec{y}} = 1$  and  $(\exists X[H])_{\vec{y}} = 0$ . Then there exists an assignment  $\vec{x}$  to X such that  $(\vec{x},\vec{y})$  satisfies F. Since every clause of H is in F, formula H is also satisfied by  $\vec{p}$ . So we have a contradiction.
- $(\exists X[F])_{\vec{y}} = 0$  and  $(\exists X[H])_{\vec{y}} = 1$ . Then there exists an assignment  $\vec{x}$  to variables of X such that  $(\vec{x},\vec{y})$  satisfies H. Since  $F_{\vec{y}} \equiv 0$ , point  $(\vec{x},\vec{y})$  falsifies F. Since  $H(\vec{p}) = 1$ ,  $(\vec{x},\vec{y})$  is a Z-boundary point of F that falsifies only clauses of G. Since  $F_{\vec{y}} \equiv 0$ ,  $(\vec{x},\vec{y})$  is an X-removable Z-boundary point of F, which leads to a contradiction.  $\square$

# Propositions of Section 4: Quantifier Elimination By Branching

**Proposition 3.** Let  $\exists X[F(X,Y)]$  be an  $\exists \text{CNF}$  and  $\vec{q}$  be an assignment to Vars(F). Let  $\vec{p}$  be a Z-boundary point of F where  $\vec{q} \subseteq \vec{p}$  and  $Z \subseteq X$ . Then if  $\vec{p}$  is removable in  $\exists X[F]$  it is also removable in  $\exists X[F_{\vec{q}}]$ .

Proof: Assume the contrary. That is  $\vec{p}$  is removable in  $\exists X[F]$  but is not removable in  $\exists X[F_{\vec{q}}]$ . The fact that  $\vec{p}$  is removable in  $\exists X[F]$  means that there is a clause C implied by F and falsified by  $\vec{p}$  that consists only of variables of Y. Since  $\vec{p}$  is not removable in  $\exists X[F_{\vec{q}}]$ , from Proposition 1 it follows that an assignment  $\vec{s}$  satisfying  $F_{\vec{q}}$  can be obtained from  $\vec{p}$  by changing only values of variables of  $X \setminus Vars(\vec{q})$ . By construction,  $\vec{p}$  and  $\vec{s}$  have identical assignments to variables of Y. So  $\vec{s}$  has to falsify C. On the other hand, by construction,  $\vec{q} \subseteq \vec{s}$ . So, the fact that  $\vec{s}$  satisfies  $F_{\vec{q}}$  implies that  $\vec{s}$  satisfies F too. Since  $\vec{s}$  falsifies C and satisfies F, clause C is not implied by F and we have a contradiction.  $\square$ 

**Proposition 4.** Let  $\exists X[F(X,Y)]$  be an  $\exists CNF$  and  $H \subset F^X$  be redundant in  $\exists X[F]$ . Let an X-clause C of  $F \setminus H$  be redundant in  $\exists X[F \setminus H]$ . Then  $H \cup \{C\}$  is redundant in  $\exists X[F]$ .

*Proof:* Denote  $H \cup \{C\}$  as H'. Assume the contrary, i.e. H' is not redundant in  $\exists X[F]$ . Then, from Proposition 2, it follows that F has an X-removable boundary  $\vec{p}$  such that every clause falsified by  $\vec{p}$  is in H'. Denote as H'' the subset of clauses of H' falsified by  $\vec{p}$ . Let us consider the two possible cases.

- Clause C is not in H''. In this case, H'' is a subset of H and the existence of  $\vec{p}$  means that H is not redundant in  $\exists X[F]$ . So we have a contradiction.
- Clause C is in H''. Redundancy of C in  $\exists X[F \setminus H]$  means that one can turn  $\vec{p}$  into an assignment  $\vec{s}$  satisfying  $F \setminus H$  by flipping values of variables from X. Since  $\vec{p}$  is X-removable,  $\vec{s}$  falsifies F. The only clauses of F falsified by  $\vec{s}$  are those of H. Since  $\vec{p}$  and  $\vec{s}$  have identical values assigned to Y, point  $\vec{s}$  is X-removable as well. Then the existence of  $\vec{s}$  means that H is not redundant in  $\exists X[F]$  and we have a contradiction.  $\Box$

#### Propositions of Section 5: Virtual Redundancy

**Proposition 5.** Let  $\vec{q}$  be an assignment to Vars(F) and clause  $C_{\vec{q}}$  be redundant in  $\exists X[F_{\vec{q}}]$ . Then, for every  $\vec{r}$  such that  $\vec{q} \subset \vec{r}$ , clause  $C_{\vec{r}}$  is virtually redundant in  $\exists X[F_{\vec{r}}]$ .

*Proof:* Let a point  $\vec{p}$  of F falsify only  $C_{\vec{r}}$  and be X-removable. Then  $\vec{p}$  falsifies only clause  $C_{\vec{q}}$  of  $F_{\vec{q}}$ . Since  $C_{\vec{q}}$  is redundant in  $\exists X[F_{\vec{q}}]$ , point  $\vec{p}$  is not X-removable in  $F_{\vec{q}}$ .  $\square$ 

## Propositions of Section 7: Atomic D-sequents

**Proposition 6.** Let  $\exists X[F]$  be an  $\exists CNF$ . Let  $v \in Vars(C)$  and by assigning value b to v where  $b \in \{0,1\}$  one satisfies C. Then D-sequent  $(v = b) \to C$  holds We will refer to it as a **D-sequent of the first kind**.

*Proof:* Let  $\vec{q}$  denote (v=b). Let W be a member formula for the D-sequent  $(v=b) \to C$ . Hence, W contains C. Clause  $C_{\vec{q}}$  is true and so is redundant in  $W_{\vec{q}}$  and hence, in  $\exists X[W_{\vec{q}}]$ .  $\Box$ 

**Proposition 7.** Let  $\exists X[F]$  be an  $\exists \text{CNF}$  formula and  $\vec{q}$  be an assignment to Vars(F). Let B, C be two clauses of F. Let  $C_{\vec{q}}$  be an X-clause and  $B_{\vec{q}}$  imply  $C_{\vec{q}}$  (i.e. every literal of  $B_{\vec{q}}$  is in  $C_{\vec{q}}$ ). Then the D-sequent  $(\vec{q}, H) \to C$  holds where  $H = \{B\}$  if  $B_{\vec{q}}$  is an X-clause and  $H = \emptyset$  otherwise. We will refer to the D-sequent above as an atomic D-sequent of the **second kind**.

*Proof:* Let W be a member formula for the D-sequent  $(\vec{q}, H) \to C$ . Then  $\{B, C\} \subseteq W$ . Indeed, if  $B_{\vec{q}}$  is an X-clause then  $B \in H$  and  $H \subseteq W$ . Otherwise, B is in W because W is obtained from F by removing only clauses that are X-clauses in subspace  $\vec{q}$ . Since  $B_{\vec{q}}$  implies  $C_{\vec{q}}$ , then  $W_{\vec{q}} \equiv W_{\vec{q}} \setminus \{C_{\vec{q}}\}$ .  $\square$ 

**Lemma 1.** Let  $\exists X[F]$  be an  $\exists CNF$  formula and  $\vec{q}$  be an assignment to Vars(F). Let C be an X-clause of F not satisfied by  $\vec{q}$  and  $v \in X$  be a variable of C such that  $v \notin Vars(\vec{q})$ . Let clause  $C_{\vec{q}}$  be blocked at v in  $F_{\vec{q}}$ . Then  $C_{\vec{q}}$  is redundant in  $\exists X[F_{\vec{q}}]$ .

*Proof:* Assume the contrary i.e.  $C_{\vec{q}}$  is not redundant in  $\exists X[F_{\vec{q}}]$ . Then there is a Z-boundary point  $\vec{p}$  where  $Z \subseteq X$  that falsifies only  $C_{\vec{q}}$  and is X-removable in  $F_{\vec{q}}$ . Let  $\vec{p'}$  be the point obtained from  $\vec{p}$  by flipping the value of v. Consider the following two possibilities.

- $\vec{p}'$  satisfies  $F_{\vec{q}}$ . Then  $\vec{p}$  is not X-removable and we have a contradiction.
- $\vec{p'}$  falsifies a clause  $C'_{\vec{q}}$  of  $F_{\vec{q}}$ . Then  $C_{\vec{q}}$  and  $C'_{\vec{q}}$  are resolvable on variable v and we have a contradiction again.  $\square$

**Proposition 8.** Let  $\exists X[F]$  be an  $\exists CNF$  formula. Let C be an X-clause of F and  $v \in Vars(C) \cap X$ . Let  $C_1, \ldots, C_k$  be the clauses of F that can be resolved with C on variable v. Let  $(\vec{q}_1, H_1) \to C_1, \ldots, (\vec{q}_k, H_k) \to C_k$  be a consistent set of D-sequents. Then D-sequent  $(q, H) \to C$  holds where  $\vec{q} = \bigcup_{i=1}^{i=k} \vec{q}_i$  and  $H = \bigcup_{i=1}^{i=k} H_i$ . We will refer to it as an atomic D-sequent of the **third kind**.

Proof: Let W be a member formula for the D-sequent  $(q, H) \to C$ . Since  $H_i \subseteq H$  holds, W is a member formula for D-sequent  $(\vec{q}_i, H_i) \to C_i$  too. So  $C_i$  is redundant in  $\exists X[W]$  in subspace  $\vec{q}_i$ . Since  $\vec{q}_i \subseteq \vec{q}$ , from Proposition 5 it follows, that  $C_i$  is redundant in  $\exists X[W]$  in subspace  $\vec{q}$  too. Since D-sequents  $(\vec{q}_1, H_1) \to C_1, \ldots, (\vec{q}_k, H_k) \to C_k$  are consistent, the clauses  $C_1, \ldots, C_k$  are redundant together in  $\exists X[W]$  in subspace  $\vec{q}$ . So clause C is blocked in  $\exists X[W]$  at variable v in subspace  $\vec{q}$ . From Lemma 1 it follows that C is redundant in  $\exists X[W]$  in subspace  $\vec{q}$ .  $\Box$ 

## Propositions of Section 8: Join Operation

**Proposition 9.** Let  $\exists X[F]$  be an  $\exists CNF$  formula. Let D-sequents  $(\vec{q'}, H') \to C$  and  $(\vec{q''}, H'') \to C$  hold. Let  $\vec{q'}$ ,  $\vec{q''}$  be resolvable on  $v \in Vars(F)$  and  $\vec{q}$  be the resolvent of  $\vec{q'}$  and  $\vec{q''}$ . Let  $H = H' \cup H''$ . Then the D-sequent  $(\vec{q}, H) \to C$  holds.

*Proof:* Denote by S' and S the D-sequents  $(\vec{q'}, H') \to C$  and  $(\vec{q}, H) \to C$  respectively. Assume that S does not hold. Then there is a member formula W of S such that C is not redundant in  $\exists X[W_{\vec{q}}]$  even virtually. This means that

- the set B of Definition 11 is not empty and
- there is a point  $\vec{p} \in B$  that is X-removable in every formula  $W_{\vec{r}}$  where  $\vec{q^*} \subseteq \vec{r} \subset \vec{q}$  (see Definition 11).

Assume for the sake of clarity that  $\vec{p}$  has the same assignment to v as  $\vec{q'}$ . Note that, since  $H' \subseteq H$ , W is a member formula of the D-sequent  $(\vec{q'}, H') \to C$ . So,  $C_{\vec{q'}}$  is redundant in  $\exists X[W_{\vec{q'}}]$ . Consider the following possibilities.

- Point  $\vec{p}$  is not X-removable in  $W_{\vec{q'}}$  and  $v \in X$ . Then it is not X-removable in  $W_{\vec{s}}$  where  $\vec{s}$  is obtained from  $\vec{q'}$  by dropping the assignment to v. Since  $\vec{s} \subset \vec{q}$  holds, this contradicts the fact that  $\vec{p}$  has to be X-removable in every subspace  $\vec{q^*} \subseteq \vec{r} \subset \vec{q}$ .
- Point  $\vec{p}$  is not X-removable in  $W_{\vec{q'}}$  and  $v \notin X$ . Then  $\vec{p}$  is not X-removable in  $W_{\vec{q}}$ . This contradicts the fact that  $\vec{p} \in B$ .
- Point  $\vec{p}$  is X-removable in  $W_{\vec{q'}}$  and it is X-removable in  $W_{\vec{r}}$  for every  $\vec{q'}^* \subseteq \vec{r} \subset \vec{q'}$ . Then  $C_{\vec{q'}}$  is not redundant in  $\exists X[W_{\vec{q'}}]$ , which contradicts the fact that D-sequent S' holds.
- Point  $\vec{p}$  is X-removable in  $W_{\vec{q'}}$ , and  $v \in X$  and  $\vec{p}$  is not X-removable in  $W_{\vec{r}}$  where  $\vec{q'}^* \subseteq \vec{r} \subset \vec{q'}$ . Then it is not X-removable in  $W_{\vec{r}^*}$  obtained from  $\vec{r}$  by dropping the assignment to v, if any (regardless of whether or not v is in X). Since  $\vec{r^*} \subset \vec{q}$  holds, this contradicts the fact that  $\vec{p}$  has to be X-removable in every subspace  $\vec{q^*} \subseteq \vec{r} \subset \vec{q}$ .  $\square$

## Propositions of Section 9: Re-usability of D-sequents

**Proposition 10.** Let  $\exists X[F]$  be an  $\exists CNF$ . Let  $(\vec{q}_1, H_1) \to C_1, \ldots, (\vec{q}_k, H_k) \to C_k$  be a consistent set of D-sequents. Then these D-sequents are composable and hence clauses  $\{C_1, \ldots, C_k\}$  are collectively redundant in  $\exists X[F]$  in subspace  $\vec{q}$  where  $\vec{q} = \bigcup_{i=1}^{i=k} \vec{q}_i$ .

*Proof:* Since  $\vec{q}_i \subseteq \vec{q}$ ,  $i=1,\ldots,k$ , from Proposition 5 it follows that D-sequents  $(\vec{q},H_i) \to C_i$ ,  $i=1,\ldots,k$  hold. Assume for the sake of simplicity that  $\pi(C_i) < \pi(C_j)$  if i < j. Then one can prove redundancy of clauses  $C_i$ ,  $i=1,\ldots,k$  in subspace  $\vec{q}$  in the order they are numbered. Denote  $(\vec{q},H_i) \to C_i$  as  $S_i$ ,  $i=1,\ldots,k$ 

 $1, \ldots, k$ . Denote by  $F_{i+1}$  formula  $F \setminus \{C_1, \ldots, C_i\}$ ,  $i = 1, \ldots, k$ . Formula  $F_1$  is set to F. Since  $S_1$  is applicable to  $F_1$  one can remove  $C_1$  in subspace  $\vec{q}$  producing formula  $F_2$ . Note that  $F_2$  is a member formula for  $S_2$ . By applying  $S_2$  to  $F_1$  one removes clause  $C_2$  in subspace  $\vec{q}$  producing  $F_3$ , a member formula for  $S_3$ . Going on in such a manner one eventually produces formula  $F_{k+1}$  thus showing that  $\{C_1, \ldots, C_k\}$  are redundant in  $\exists X[F]$  in subspace  $\vec{q}$ .  $\Box$ 

## Propositions of Section 10: Two Useful Transformations Of D-sequents

**Lemma 2.** Let  $\exists X[F(X,Y)]$  be an  $\exists \text{CNF}$  formula. Let  $\vec{q}$  be an assignment to Vars(F). Let C be an X-clause redundant in  $\exists X[F_{\vec{q}}]$ . Let C be also redundant in formula  $\exists X[W_{\vec{q}}]$  where W is obtained from F by dropping X-clauses that are redundant in  $\exists X[F_{\vec{q}}]$ . Then every point  $\vec{p}$  falsifying only clause  $C_{\vec{q}}$  of  $W_{\vec{q}}$  can be turned into a point satisfying F by changing values of (some) variables of X.

*Proof:* Denote by  $C_1, \ldots, C_k$  the X-clauses dropped from F to obtain W. Assume that these clauses were dropped in the numbering order. That is clause  $C_i$  is redundant in  $\exists X[W_i]$  in subspace  $\vec{q}$  where  $W_1 = F$  and  $W_{i+1} = W_i \setminus \{C_i\}$ ,  $i = 1, \ldots, k$ . So  $W_{k+1} = F \setminus \{C_1, \ldots, C_k\} = W$ .

Let  $\vec{p}$  be a point falsifying only C of W in subspace  $\vec{q}$ . Since clause C is redundant, one can turn  $\vec{p}$  into a point satisfying W in subspace  $\vec{q}$  by changing values of variables of X. Denote the new point as  $\vec{p}$  again. If  $\vec{p}$  satisfies F then we are done. Otherwise,  $\vec{p}$  falsifies some clauses  $C_1, \ldots, C_k$ .

Let  $C_i$  be the clause with the largest index that is falsified by  $\vec{p}$ . Note that  $\vec{p}$  falsifies only clause  $C_i$  in  $W_i$  in subspace  $\vec{q}$ . Since  $C_i$  is redundant in  $\exists X[W_i]$  in subspace  $\vec{q}$ , one can turn  $\vec{p}$  into an assignment satisfying  $W_i$  in subspace  $\vec{q}$  by changing only assignments to X. Denote the new point as  $\vec{p}$  again. If  $\vec{p}$  satisfies F we are done. Otherwise,  $\vec{p}$  falsifies some clauses  $C_1, \ldots, C_{i-1}$ .

Going on in such a manner one eventually builds a point satisfying F that is obtained from the very first point  $\vec{p}$  by changing only assignments to variables of X.  $\square$ 

**Proposition 11.** Let D-sequent  $(\exists X[F], \vec{q}, H) \to C$  hold and R be a CNF formula implied by F. Then D-sequent  $(\exists X[F \land R], \vec{q}, H) \to C$  holds too.

Proof: Let  $W^R$  be a member formula for  $(\exists X[F \land R], \vec{q}, H) \to C$ . Denote by W the formula  $W^R \setminus R$ . Note that W is a member formula for  $(\exists X[F], \vec{q}, H) \to C$ . If  $W = W^R$ , then C is redundant in  $\exists X[W]$  (and hence in  $\exists X[W^R]$ ) in subspace  $\vec{q}$ . Now consider the case  $W \subset W^R$ . Assume that C is not redundant in  $\exists X[W^R]$  in subspace  $\vec{q}$ . Then there is a point  $\vec{p}$  such that

- $\vec{p}$  falsifies only clause C of  $W_R$  in subspace  $\vec{q}$  where  $\vec{q} \subseteq \vec{p}$
- $\vec{p}$  is X-removable for every  $\vec{r}$  obtained from  $\vec{q}$  by dropping (some) assignments to X.

Since C is redundant in  $\exists X[W]$  in subspace  $\vec{q}$ , from Lemma 2, it follows that one can turn  $\vec{p}$  into point  $\vec{p^*}$  satisfying F by changing only values of variables of X. Since R is implied by F,  $\vec{p^*}$  satisfies  $F \land R$  as well. Hence  $\vec{p^*}$  satisfies  $W_R$  and  $\vec{p}$  is not X-removable in subspace  $\vec{r}$  obtained by dropping from  $\vec{q}$  all assignments to X. So we have a contradiction.  $\square$ 

**Lemma 3.** Let  $\exists X[F]$  be an  $\exists CNF$  formula and  $\vec{q}$  be an assignment to Vars(F). Let C' and C'' be X-clauses of F and  $\{C', C''\}$  be redundant in  $\exists X[F]$  in subspace  $\vec{q}$ . Then clause C' is redundant in  $\exists X[F \setminus \{C''\}]$  in subspace  $\vec{q}$ .

*Proof:* Assume that C' is not redundant in  $\exists X[F \setminus \{C''\}]$ . in subspace  $\vec{q}$ . Then there is a point  $\vec{p}$  falsifying only  $C'_{\vec{q}}$  that is X-removable in every subspace  $\vec{q^*} \subseteq \vec{r} \subseteq \vec{q}$  (see Definition 11). Then, from Proposition 2 it follows that clauses  $\{C', C''\}$  are not redundant in  $\exists X[F]$  in subspace  $\vec{q}$  (even virtually).  $\Box$ 

**Lemma 4.** Let  $\exists X[F]$  be an  $\exists \text{CNF.}$  Let D-sequents  $(\vec{q'}, H') \to C'$  and  $(\vec{q''}, H'') \to C''$  hold where  $\vec{q''} \subseteq \vec{q'}$  and  $H'' \subset H'$  and  $C'' \in H'$  and  $C' \notin H''$ . Then the D-sequent  $(\vec{q'}, H' \setminus \{C''\}) \to C'$  holds.

Proof: Let W be a member formula for  $(\vec{q'}, H' \setminus \{C'''\}) \to C'$ . Let us show that C' is redundant in  $\exists X[W]$  in subspace  $\vec{q'}$  and so  $(\vec{q'}, H' \setminus \{C'''\}) \to C'$  holds. Consider the following two situations. First, assume that clause  $C'' \in W$ . Then W is a member formula for the D-sequent  $(\vec{q'}, H') \to C'$  and hence C' is redundant in  $\exists X[W]$  in subspace  $\vec{q'}$ .

Now assume that clause  $C'' \notin W$ . Denote D-sequents  $(\vec{q'}, H') \to C'$  and  $(\vec{q'}, H'') \to C''$  (computed with respect to  $\exists X[F]$ ) as S' and S'' respectively. Denote formula  $W \cup \{C''\}$  by W''. Note that W'' is a member formula for S' and S''. Besides, S' and S'' are consistent. So, in particular, S'' can be used after S'. By applying S' and S'' one shows that  $\{C', C''\}$  are redundant in  $\exists X[W'']$  in subspace  $\vec{q'}$ . From Lemma 3 it follows that C' is redundant in  $\exists X[W]$  in subspace  $\vec{q'}$ .  $\Box$ 

**Lemma 5.** Let  $\exists X[F]$  be an  $\exists CNF$ . Let D-sequent  $(\vec{q}, H) \to C$  hold. Let G be an arbitrary subset of X-clauses of  $F \setminus H$ . Then D-sequent  $(\vec{q}, H \cup G) \to C$  holds too.

*Proof:* Let W be a member formula for  $(\vec{q}, H \cup G) \to C$ . Then W is a member formula for  $(\vec{q}, H) \to C$ . So, C is redundant in  $\exists X[W]$  in subspace  $\vec{q}$ .  $\Box$ 

**Proposition 12.** Let  $\exists X[F]$  be an  $\exists CNF$ . Let  $(\vec{q'}, H') \to C'$  and  $(\vec{q''}, H'') \to C''$  be two D-sequents forming a consistent set (see Definition 19). Let C'' be in H'. Then D-sequent  $(\vec{q}, H) \to C'$  holds where  $\vec{q} = \vec{q'} \cup \vec{q''}$  and  $H = (H' \setminus \{C''\}) \cup H''$ .

*Proof:* From Proposition 5 it follows that D-sequent  $(\vec{q}, H') \to C'$  holds. From Lemma 5 it follows that  $(\vec{q}, H' \cup H'') \to C'$  holds too. The latter and D-sequent  $(\vec{q''}, H'') \to C''$  satisfy the conditions of Lemma 4. (Note that H'' cannot contain C' because  $(\vec{q'}, H') \to C'$  and  $(\vec{q''}, H'') \to C''$  are consistent.) This entails that  $(\vec{q}, H) \to C'$  holds.  $\square$ 

**Proposition 13.** Let  $\exists X[F]$  be an  $\exists CNF$  and  $(\vec{q}_1, H_1) \to C_1, \ldots, (\vec{q}_k, H_k) \to C_k$ be consistent D-sequents where  $H_i \subseteq \{C_1, \ldots, C_k\}, i = 1, \ldots, k$ . Assume, for the sake of simplicity, that the numbering order is consistent with the order constraints. Let  $C_m$  be in  $H_i$ . Then, by repeatedly applying the transformation of Proposition 12, one can produce D-sequent  $(\vec{q}, H_i \setminus \{C_m\}) \rightarrow C_i$  where

$$\vec{q}_i \subseteq \vec{q} \subseteq \vec{q}_i \cup \bigcup_{j=m}^{j=k} \vec{q}_j.$$

*Proof:* Let  $S_j$  denote D-sequent  $(\vec{q}_j, H_j) \rightarrow C_j, j = 1, \ldots, k$ . Note that i < jm holds, otherwise,  $S_m$  would be used before  $S_i$  proving  $C_m$  redundant and thus making  $S_i$  inapplicable. Let us use Proposition 12 to remove clause  $C_m$ from the order constraint of  $S_m$ . This produces a new D-sequent S equal to  $(\vec{q}, (H_i \cup H_m) \setminus \{C_m\}) \to C_i$  where  $\vec{q} = \vec{q}_i \cup \vec{q}_m$ . If  $H_m \subseteq H_i$  holds, the proposition in question is proved. Otherwise, one keeps removing clauses from the order constraint of D-sequent S.

Let  $C_r$  be the clause of  $H_m \setminus H_i$  with the largest index. Note that m < rholds (and, hence, i < m < r) for the same reason i < m does. By applying Proposition 12 to remove clause  $C_r$  from the order constraint of S one produces a new D-sequent S equal to  $(\vec{q}, (H_i \cup H_m \cup H_r) \setminus \{C_m, C_r\}) \rightarrow C_i$  where  $\vec{q} =$  $\vec{q_i} \cup \vec{q_m} \cup \vec{q_r}$ . Note that since i < r and m < r, set  $H_r$  cannot contain  $C_m$  or  $C_r$ . If  $(H_m \cup H_r) \setminus \{C_r\}$  is a subset of  $H_i$  the proposition in question is proved. Otherwise, one picks the clause of  $(H_m \cup H_r) \setminus \{C_r\}$  with the largest index that is not in  $H_i$  and removes it by applying the transformation of Proposition 12.

The procedure above goes one until one produces a D-sequent S with order constraint  $H_i \setminus \{C_m\}$ . This procedure converges, since one always removes a clause with the largest index and so this clause cannot re-appear in the order constraint of S. Thus, eventually, in no more than k-m steps, all the clauses that are not in  $H_i \setminus \{C_m\}$  will be removed from the order constraint of S.  $\square$