

Structure-Aware Computing By Partial Quantifier Elimination

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Abstract. By structure-aware computing (SAC) we mean computing that is formula-specific i.e., takes into account the structure of the formula at hand. Virtually all efficient algorithms of hardware verification employ some form of SAC. We relate SAC to *partial quantifier elimination* (PQE). The latter is a generalization of regular quantifier elimination where one can take a *part* of the formula out of the scope of quantifiers. The objective of this paper is to emphasize the significance of studying PQE for enhancing the *existing* methods of SAC and creating *new* ones. First, we show that interpolation (that can be viewed as an instance of SAC) is a special case of PQE. Then we describe application of SAC by PQE to three different problems of hardware verification: property generation, equivalence checking and model checking. Besides, we discuss using SAC by PQE for SAT solving.

1 Introduction

Arguably, almost all efficient algorithms of hardware verification take into account the structure of the formula at hand i.e., they are formula-specific (see Appendix A). We will say that those algorithms are based on *structure-aware computing*. In this paper, we relate structure-aware computing to partial quantifier elimination (PQE). Our objective here is to show the importance of studying PQE for designing efficient structure-aware algorithms.

In this paper, we consider only *propositional* formulas in *conjunctive normal form* (CNF) and only *existential* quantifiers. PQE is a generalization of regular quantifier elimination (QE) that is defined as follows [1]. Let $F(X, Y)$ be a quantifier-free formula where X, Y are sets of variables and G be a subset of clauses¹ of F . Given a formula $\exists X[F]$, the PQE problem is to find a quantifier-free formula $H(Y)$ such that $\exists X[F] \equiv H \wedge \exists X[F \setminus G]$. In contrast to *full* QE, only the clauses of G are taken out of the scope of quantifiers hence the name *partial* QE. We will refer to H as a *solution* to PQE. Note that QE is just a special case of PQE where $G = F$ and the entire formula is unquantified. A key role in PQE solving is played by *redundancy based reasoning*: to take a set of clauses G out of $\exists X[F(X, Y)]$, one essentially needs to find a formula $H(Y)$ that

¹ Given a CNF formula F represented as the conjunction of clauses $C_0 \wedge \dots \wedge C_k$, we will also consider F as the *set* of clauses $\{C_0, \dots, C_k\}$.

makes G *redundant* in $H \wedge \exists X[F]$. The appeal of PQE is that it can be *much more efficient* than QE if G is a small piece of F : to solve PQE one needs to make redundant only G . (In QE, one has to make redundant the *entire* formula F .) So, it is beneficial to design algorithms based on PQE.

The idea of structure-aware computing by PQE is derived from the following observation. QE is a *semantic* operation in the sense that if $\exists X[F(X, Y)] \equiv H(Y)$ and $F' \equiv F$, then $\exists X[F'] \equiv H$. That is, the truth table of H depends only on the truth table of F . On the other hand, PQE is a *structural* (i.e., formula-specific) operation in the following sense. Given $F(X, Y), H(Y)$, and $G \subseteq F$, the fact that $\exists X[F] \equiv H \wedge \exists X[F \setminus G]$ and $F' \equiv F$ **does not** imply that $\exists X[F'] \equiv H \wedge \exists X[F' \setminus G]$. In other words, H cannot be computed using the truth table of F alone. So, H depends on (and hence PQE can exploit) the *specifics* of F . In this sense, PQE is similar to interpolation that is a structural operation *too*. As we show in Section 4, interpolation is a special case of PQE.

In addition to relating interpolation to PQE and, more generally, structure-aware computing to PQE, the *contribution* of this paper is as follows. We demonstrate structure-aware computing by PQE using various problems of hardware verification. In Section 5, we consider our previous results on applying PQE to property generation (a generalization of testing) and equivalence checking [2,3] *in the context* of structure-aware computing. We want to show here that PQE can drastically enhance the *existing* methods of structure-aware computing successfully applied in testing and equivalence checking. In Sections 6 and 7, we employ model checking and SAT to demonstrate that PQE can also create *new* methods of structure-aware computing.

The main body of this paper is structured as follows. (Some additional information can be found in the appendix.) In Section 2, we give basic definitions. A high-level view of PQE solving and some examples are presented in Section 3. As mentioned above, Sections 4-7 relate interpolation and structure-aware computing to PQE. In Section 8, we make conclusions.

2 Basic Definitions

In this section, when we say “formula” without mentioning quantifiers, we mean “a quantifier-free formula”.

Definition 1. *We assume that formulas have only Boolean variables. A **literal** of a variable v is either v or its negation. A **clause** is a disjunction of literals. A formula F is in conjunctive normal form (**CNF**) if $F = C_0 \wedge \dots \wedge C_k$ where C_0, \dots, C_k are clauses. We will also view F as the **set of clauses** $\{C_0, \dots, C_k\}$. We assume that **every formula is in CNF** unless otherwise stated.*

Definition 2. *Let F be a formula. Then $\mathbf{Vars}(F)$ denotes the set of variables of F and $\mathbf{Vars}(\exists X[F])$ denotes $\mathbf{Vars}(F) \setminus X$.*

Definition 3. *Let V be a set of variables. An **assignment** \vec{q} to V is a mapping $V' \rightarrow \{0, 1\}$ where $V' \subseteq V$. We will denote the set of variables assigned in \vec{q} as*

$\text{Vars}(\vec{q})$. We will refer to \vec{q} as a **full assignment** to V if $\text{Vars}(\vec{q}) = V$. We will denote as $\vec{q} \subseteq \vec{r}$ the fact that a) $\text{Vars}(\vec{q}) \subseteq \text{Vars}(\vec{r})$ and b) every variable of $\text{Vars}(\vec{q})$ has the same value in \vec{q} and \vec{r} .

Definition 4. A literal and a clause are said to be **satisfied** (respectively **falsified**) by an assignment \vec{q} if they evaluate to 1 (respectively 0) under \vec{q} .

Definition 5. Let C be a clause. Let H be a formula that may have quantifiers, and \vec{q} be an assignment to $\text{Vars}(H)$. If C is satisfied by \vec{q} , then $C_{\vec{q}} \equiv \mathbf{1}$. Otherwise, $C_{\vec{q}}$ is the clause obtained from C by removing all literals falsified by \vec{q} . Denote by $H_{\vec{q}}$ the formula obtained from H by removing the clauses satisfied by \vec{q} and replacing every clause C unsatisfied by \vec{q} with $C_{\vec{q}}$.

Definition 6. Let G, H be formulas that may have existential quantifiers. We say that G, H are **equivalent**, written $G \equiv H$, if $G_{\vec{q}} = H_{\vec{q}}$ for all full assignments \vec{q} to $\text{Vars}(G) \cup \text{Vars}(H)$.

Definition 7. Let $F(X, Y)$ be a formula and $G \subseteq F$ and $G \neq \emptyset$. The clauses of G are said to be **redundant in $\exists X[F]$** if $\exists X[F] \equiv \exists X[F \setminus G]$. If $F \setminus G$ implies G , the clauses of G are **redundant in $\exists X[F]$** but the opposite is not true.

Definition 8. Given a formula $\exists X[F(X, Y)]$ and G where $G \subseteq F$, the **Partial Quantifier Elimination (PQE)** problem is to find $H(Y)$ such that $\exists X[F] \equiv H \wedge \exists X[F \setminus G]$. (So, PQE takes G out of the scope of quantifiers.) The formula H is called a **solution** to PQE. The case of PQE where $G = F$ is called **Quantifier Elimination (QE)**.

Example 1. Consider formula $F = C_0 \wedge \dots \wedge C_4$ where $C_0 = \bar{x}_2 \vee x_3$, $C_1 = y_0 \vee x_2$, $C_2 = y_0 \vee \bar{x}_3$, $C_3 = y_1 \vee x_3$, $C_4 = y_1 \vee \bar{x}_3$. Let $Y = \{y_0, y_1\}$ and $X = \{x_2, x_3\}$. Consider the PQE problem of taking C_0 out of $\exists X[F]$ i.e., finding $H(Y)$ such that $\exists X[F] \equiv H \wedge \exists X[F \setminus \{C_0\}]$. One can show that $\exists X[F] \equiv y_0 \wedge \exists X[F \setminus \{C_0\}]$ (see Subsection 3.3) i.e., $H = y_0$ is a solution to this PQE problem.

Definition 9. Given a formula $\exists X[F(X, Y)]$ and G where $G \subseteq F$, the **decision version of PQE** is to check if G is redundant in $\exists X[F]$ i.e., if $\exists X[F] \equiv \exists X[F \setminus G]$.

Definition 10. Let clauses C', C'' have opposite literals of exactly one variable $w \in \text{Vars}(C') \cap \text{Vars}(C'')$. Then C', C'' are called **resolvable** on w .

Definition 11. Let C be a clause of a formula F and $w \in \text{Vars}(C)$. The clause C is said to be **blocked** [4] in F with respect to the variable w if no clause of F is resolvable with C on w .

Proposition 1. Let a clause C be blocked in a formula $F(X, Y)$ with respect to a variable $x \in X$. Then C is redundant in $\exists X[F]$, i.e., $\exists X[F \setminus \{C\}] \equiv \exists X[F]$.

The proofs of propositions are given in Appendix B.

3 PQE solving

In this section, we briefly describe the PQE algorithm called *DS-PQE* [1]. Our objective here is just to give an idea of how the PQE problem can be solved. So, in Subsection 3.2, we give a high-level description of this algorithm. Subsections 3.3 and 3.4 provide examples of PQE solving.

3.1 Some background

Information on QE in propositional logic can be found in [5,6,7,8,9,10]. QE by redundancy based reasoning is presented in [11,12]. One of the merits of such reasoning is that it allows to introduce *partial* QE. A description of PQE algorithms and their sources can be found in [1,3,13,14,15].

3.2 High-level view

Like all existing PQE algorithms, *DS-PQE* uses *redundancy based reasoning* justified by the proposition below.

Proposition 2. *Formula $H(Y)$ is a solution to the PQE problem of taking G out of $\exists X[F(X, Y)]$ (i.e., $\exists X[F] \equiv H \wedge \exists X[F \setminus G]$) iff*

1. $F \Rightarrow H$ and
2. $H \wedge \exists X[F] \equiv H \wedge \exists X[F \setminus G]$

Thus, to take G out of $\exists X[F(X, Y)]$, it suffices to find a formula $H(Y)$ implied by F that makes G *redundant* in $H \wedge \exists X[F]$. We will refer to the clauses of G as **target** ones.

Below, we provide some basic facts about *DS-PQE*. Since taking out an unquantified clause is trivial, we assume that the formula G contains only *quantified* clauses. *DS-PQE* finds a solution to the PQE problem above by branching on variables of F . The idea here is to reach a subspace \vec{q} where every clause of G can be easily proved or made redundant in $\exists X[F]$. Importantly, *DS-PQE* branches on unquantified variables, i.e., those of Y , *before* quantified ones. Like a SAT-solver, *DS-PQE* runs Boolean Constraint Propagation (BCP). If a conflict occurs in subspace \vec{q} , *DS-PQE* generates a conflict clause K and adds it to F to *make* clauses of G redundant in subspace \vec{q} . However, most frequently, proving redundancy of G in a subspace does not require a conflict.

If a target clause C becomes unit² in subspace \vec{q} , *DS-PQE* *temporarily* extends the set of target clauses G . Namely, *DS-PQE* adds to G every clause

² An unsatisfied clause is called *unit* if it has only one unassigned literal. Due to special decision making of *DS-PQE* (variables of Y are assigned before those of X), if the target clause C becomes unit, its unassigned variable is **always in X** . We assume here that C contains at least one variable of X . (Taking out a clause depending only on unquantified variables, i.e. those of Y , is trivial.)

that is resolvable with C on its only unassigned variable (denote is as x). This is done to facilitate proving redundancy of C . If the added clauses are proved redundant in subspace \vec{q} , clause C is blocked at x . Since $x \in X$, then C is redundant in subspace \vec{q} . Extending G means that *DS-PQE* may need to prove redundancy of clauses other than those of G . The difference is that every clause of the original set G must be proved redundant *globally* whereas clauses added to G need to be proved redundant only *locally* (in some subspaces).

To express the redundancy of a clause C in a subspace \vec{q} , *DS-PQE* uses a record $\vec{q} \rightarrow C$ called a **D-sequent** (see below). It states the redundancy of C in the current formula $\exists X[F]$ in subspace \vec{q} . This D-sequent also holds in any formula $\exists X[F^*]$ where F^* is obtained from F by adding clauses implied by F . A D-sequent derived for a target clause when its redundancy is easy to prove is called **atomic**. D-sequents derived in different branches can be resolved similarly to clauses³. For every target clause C of the original formula G , *DS-PQE* uses such resolution to eventually derive the D-sequent $\emptyset \rightarrow C$. The latter states the redundancy of C in the *entire* space. At this point *DS-PQE* terminates. The solution $H(Y)$ to the PQE problem found by *DS-PQE* consists of the unquantified clauses added to the initial formula F by *DS-PQE* to make G redundant.

3.3 An example of PQE solving

Here we show how *DS-PQE* solves Example 1 introduced in Section 2. Recall that one takes $G = \{C_0\}$ out of $\exists X[F(X, Y)]$ where $F = C_0 \wedge \dots \wedge C_4$ and $C_0 = \bar{x}_2 \vee x_3$, $C_1 = y_0 \vee x_2$, $C_2 = y_0 \vee \bar{x}_3$, $C_3 = y_1 \vee x_3$, $C_4 = y_1 \vee \bar{x}_3$ and $Y = \{y_0, y_1\}$ and $X = \{x_2, x_3\}$. That is, one needs to find $H(Y)$ such that $\exists X[F] \equiv H \wedge \exists X[F \setminus \{C_0\}]$.

Assume that *DS-PQE* picks the variable y_0 for branching and first explores the branch $\vec{q}' = (y_0 = 0)$. In subspace \vec{q}' , clauses C_1, C_2 become unit. After assigning $x_2 = 1$ to satisfy C_1 , the clause C_0 turns into unit too and a conflict occurs (to satisfy C_0 and C_2 , one has to assign the opposite values to x_3). After a standard conflict analysis [16], a conflict clause $K = y_0$ is obtained by resolving C_1 and C_2 with C_0 . To *make* C_0 redundant in subspace \vec{q}' , *DS-PQE* adds K to F . The redundancy of C_0 is expressed by the D-sequent $\vec{q}' \rightarrow C_0$. This D-sequent is an example of an *atomic* one. It asserts that C_0 is redundant in subspace \vec{q}' .

Having finished the first branch, *DS-PQE* considers the second branch: $\vec{q}'' = (y_0 = 1)$. Since the clause C_1 is satisfied by \vec{q}'' , no clause of F is resolvable with C_0 on variable x_2 in subspace \vec{q}'' . Hence, C_0 is blocked at variable x_2 and thus redundant in $\exists X[F]$ in subspace \vec{q}'' . So, *DS-PQE* generates the D-sequent $\vec{q}'' \rightarrow C_0$. This D-sequent is another example of an *atomic* D-sequent. It states that C_0 is *already* redundant in $\exists X[F]$ in subspace \vec{q}'' (without adding a new clause). Then *DS-PQE* resolves the D-sequents $(y_0 = 0) \rightarrow C_0$ and $(y_0 = 1) \rightarrow C_0$ (derived in the first and second branches respectively) on variable

³ In the previous papers (e.g., [12]) we called the operation of resolving D-sequents *join*.

y_0 . This resolution produces the D-sequent $\emptyset \rightarrow C_0$ stating the redundancy of C_0 in $\exists X[F]$ in the *entire* space (i.e., globally). Recall that $F_{fin} = K \wedge F_{init}$ where F_{fin} and F_{init} denote the final and initial formula F respectively. That is K is the only unquantified clause added to F_{init} . So, *DS-PQE* returns K as a solution $H(Y)$. The clause $K = y_0$ is indeed a solution since it is implied by F_{init} and adding K makes C_0 redundant in $H \wedge \exists X[F_{init}]$. So both conditions of Proposition 2 are met and thus $\exists X[F_{init}] \equiv y_0 \wedge \exists X[F_{init} \setminus \{C_0\}]$.

3.4 An example of adding temporary targets

Let $F = C_0 \wedge C_1 \wedge C_2 \wedge \dots$ where $C_0 = y_0 \vee x_1$, $C_1 = \bar{x}_1 \vee x_2 \vee x_3$, $C_2 = \bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3$. Let C_1 and C_2 be the only clauses of F with the literal \bar{x}_1 . Consider the problem of taking formula G out of $\exists X[F(X, Y)]$ where $G = \{C_0\}$ (we assume that $y_0 \in Y$ and $x_1, x_2, x_3 \in X$). Suppose *DS-PQE* explore the branch $\vec{q} = (y_0 = 0)$. In the subspace \vec{q} , the target clause C_0 turns into the unit clause x_1 . In this case, *DS-PQE* adds to the set of targets G the clauses C_1 and C_2 i.e., the clauses that are resolvable with C_0 on x_1 .

The idea here is to facilitate proving redundancy of C_0 in subspace \vec{q} . Namely, if C_1 and C_2 are proved redundant in subspace \vec{q} , the target C_0 becomes *blocked* and hence redundant in subspace \vec{q} . Clauses C_1, C_2 are added to G only in subspace \vec{q} i.e., *temporarily*. As soon as *DS-PQE* leaves this subspace, C_1, C_2 are removed from G .

4 PQE And Interpolation

Interpolation [17,18] is an example of structure-aware computing. In this section, we show that interpolation can be viewed as a special case of PQE. Let $A(X, Y) \wedge B(Y, Z)$ be an unsatisfiable formula where X, Y, Z are sets of variables. Let $I(Y)$ be a formula such that $A \wedge B \equiv I \wedge B$ and $A \Rightarrow I$. Replacing $A \wedge B$ with $I \wedge B$ is called *interpolation* and I is called an *interpolant*. PQE is similar to interpolation in the sense that the latter is a *structural* rather than semantic operation. Indeed, let $A'(X, Y) \wedge B'(Y, Z)$ be a formula such that $A' \wedge B' \equiv A \wedge B$ but $A' \not\equiv A$ and (or) $B' \not\equiv B$. Then, in general, I is *not* an interpolant of $A' \wedge B'$ i.e., $A' \wedge B' \not\equiv I \wedge B'$ and (or) $A' \not\Rightarrow I$.

Now, let us describe interpolation in terms of PQE. Consider the formula $\exists W[A \wedge B]$ where $W = X \cup Z$ and A, B are the formulas above. Let $A^*(Y)$ be obtained by taking A out of the scope of quantifiers i.e., $\exists W[A \wedge B] \equiv A^* \wedge \exists W[B]$. Since $A \wedge B$ is unsatisfiable, $A^* \wedge B$ is unsatisfiable too. So, $A \wedge B \equiv A^* \wedge B$. If $A \Rightarrow A^*$, then A^* is an interpolant. The *general case* of PQE that takes A out of $\exists W[A \wedge B]$ is different from the instance above in three aspects. First, $A \wedge B$ can be *satisfiable*. Second, one does not assume that $\text{Vars}(B) \subset \text{Vars}(A \wedge B)$. That is, in general, PQE is *not meant* to produce a new formula with a smaller set of variables. Third, a solution A^* is generally implied by $A \wedge B$ rather than by A *alone*. So, one can say that interpolation is a special case of PQE. This implies that PQE enables more *powerful* structure-aware computing than interpolation.

5 Some Previous Results As Structure-Aware Computing

In this section, we present our previous results on property generation [3] and equivalence checking [2] in terms of structure-aware computing. We describe these results by the example of combinational circuits. Our objective here is to show that using PQE allows to enhance the *existing* methods of structure-aware computing.

5.1 Representing a combinational circuit by a CNF formula

Let $M(X, V, W)$ be a combinational circuit where X, V, W are sets of internal, input, and output variables of M respectively. Let $F(X, V, W)$ denote a formula specifying M . As usual, this formula is obtained by Tseitsin’s transformations [19]. Namely, $F = F_{g_0} \wedge \dots \wedge F_{g_k}$ where g_0, \dots, g_k are the gates of M and F_{g_i} specifies the functionality of gate g_i .

Example 2. Let g be a 2-input AND gate defined as $x_2 = x_0 \wedge x_1$ where x_2 denotes the output value and x_0, x_1 denote the input values of g . Then g is specified by the formula $F_g = (\bar{x}_0 \vee \bar{x}_1 \vee x_2) \wedge (x_0 \vee \bar{x}_2) \wedge (x_1 \vee \bar{x}_2)$. Every clause of F_g is falsified by an inconsistent assignment (where the output value of g is not implied by its input values). For instance, $x_0 \vee \bar{x}_2$ is falsified by the inconsistent assignment $x_0 = 0, x_2 = 1$. So, every assignment *satisfying* F_g corresponds to a *consistent* assignment to g and vice versa. Similarly, every assignment satisfying the formula F above is a consistent assignment to the gates of M and vice versa.

5.2 Testing, property generation, and structure-aware computing

Testing is a workhorse of functional verification. The appeal of testing is that it is surprisingly effective in bug hunting taking into account that the set of generated tests makes up only a tiny part of the truth table. This effectiveness can be attributed to the fact that modern procedures are aimed at testing a particular *implementation* rather than sampling the truth table. So, testing can be viewed as an instance of structure-aware computing. In this subsection, we recall property generation by PQE [3] that is a generalization of testing. We show that using PQE dramatically boosts the power of structure-aware computing.

In addition to incompleteness, testing has the following flaw. Let $M(X, V, W)$ be a combinational circuit and F be a formula specifying M as described above. Let \vec{v} denote a single test i.e., a full assignment to V . The input/output behavior corresponding to \vec{v} can be cast as a property $H^{\vec{v}}(V, W)$ of M (i.e., F implies $H^{\vec{v}}$, see Appendix C). If the test \vec{v} exposes a bug, then $H^{\vec{v}}$ is an *unwanted* property of M . The flaw above is that $H^{\vec{v}}$ is a **weakest** property of M . So, testing can overlook a bug that gets easily exposed only by a *stronger* unwanted property. A comprehensive solution would be to generate the truth table $T(V, W)$, which is the *strongest* property of M . (T can be produced by performing QE on $\exists X[F]$ i.e., $T \equiv \exists X[F]$.) However, computing T can be prohibitively expensive. PQE allows to generate properties that are much stronger than single-test properties $H^{\vec{v}}$ but can be generated much more efficiently than the truth table T .

For the sake of simplicity, consider property generation by taking a single clause C out of $\exists X[F]$. Let $H(V, W)$ be a solution, i.e., $\exists X[F] \equiv H \wedge \exists X[F \setminus \{C\}]$. Since $F \Rightarrow H$, the solution H is a *property* of M . If H is an *unwanted* property, M has a bug. If taking out C is still too hard, one can simplify the problem by clause *splitting*. The idea here is to replace C with clauses $C \vee p$ and $C \vee \bar{p}$ where $p \in \text{Vars}(F)$ and take out, say, $C \vee p$ instead of C . Then PQE becomes simpler but produces a weaker property H . Given a single test \vec{v} , one can produce the single-test property $H^{\vec{v}}$ by combining PQE with splitting C on *all* input variables (see Appendix C).

Property generation can be viewed as an instance of **structure-aware computing**. Indeed, let M' be a circuit logically equivalent to M but having a different structure and F' be a formula specifying M' . Let H be a property obtained by taking a *single* clause out of $\exists X[F]$. Intuitively, to produce H from $\exists X[F']$, one may need to take out a *large* set of clauses. Then H is much easier to obtain (and hence more “natural”) for M than for M' . Property generation by PQE is a much more powerful method of structure-aware computing than testing because it can generate much stronger properties ranging from $H^{\vec{v}}$ to T .

To give an idea about the status quo, we describe here some experimental results on property generation reported in [3]. Those results were obtained by an optimized version of *DS-PQE*. The latter was used to generate properties for the combinational circuit M_k obtained by unfolding a sequential circuit N for k time frames. Those properties were employed to generate invariants of N . A sample of HWMCC benchmarks containing from 100 to 8,000 latches was used in those experiments. With the time limit of 10 seconds, *DS-PQE* managed to generate a lot of properties of M_k that turned out to be invariants of N . *DS-PQE* also successfully generated an *unwanted* invariant of a tailor-made FIFO buffer and so identified a hard-to-find bug.

5.3 Equivalence checking and structure-aware computing

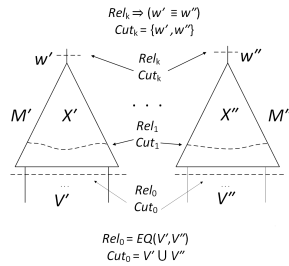


Fig. 1: Proving equivalence by the CP method

In this subsection, we discuss equivalence checking by PQE [2] in the context of structure-aware computing. Let $M'(X', V', w')$ and $M''(X'', V'', w'')$ be the single-output combinational circuits to check for equivalence. Here X^α, V^α are the sets of internal and input variables and w^α is the output variable of M^α where $\alpha \in \{', ''\}$. Circuits M', M'' are called *equivalent* if they produce identical values of w', w'' for identical inputs \vec{v}', \vec{v}'' (i.e., identical full assignments to V', V'').

Let $F = F' \wedge F''$ where $F'(X', V', w')$ and $F''(X'', V'', w'')$ specify M' and M'' respectively as described in Subsection 5.1. Let

$Z = X' \cup X'' \cup V' \cup V''$. A straightforward (but hugely inefficient) method of

equivalence checking is to perform QE on $\exists Z[Eq \wedge F]$. Here $Eq(V', V'')$ is a formula such that $Eq(\vec{v}', \vec{v}'') = 1$ iff $\vec{v}' = \vec{v}''$. The most efficient existing methods of equivalence checking use the method that we will call **cut propagation (CP)** [20,21,22]. The *CP* method can be viewed as an efficient *approximation* of QE meant for proving equivalence of circuits that are *very similar*.

The idea of the *CP* method is to build a sequence of cuts Cut_0, \dots, Cut_k of M' and M'' (see Fig. 1) and find cut points of M', M'' for which some simple *pre-defined* relations Rel_0, \dots, Rel_k hold (e.g., functional equivalence). Computations move from inputs to outputs where $Cut_0 = V' \cup V''$ and $Cut_k = \{w', w''\}$. The relation Rel_0 is *set* to $Eq(V', V'')$ whereas Rel_1, \dots, Rel_k are *computed*. The objective of the *CP* method is to show that $Rel_k = (w' \equiv w'')$. The main flaw of the *CP* method is that circuits M' and M'' may not have cut points related by pre-defined relations even if M' and M'' are very similar. In this case the *CP* method fails. So, it is *incomplete* even for similar circuits M', M'' . Despite its flaws, *CP* is a successful practical method, which can be attributed to its **structure-aware computing** (exploiting the similarity of M' and M'').

In [2] we presented a method of equivalence checking employing PQE. It is based on the proposition below.

Proposition 3. *Assume M', M'' do not implement a constant (0 or 1). Assume $\exists Z[Eq \wedge F] \equiv H \wedge \exists Z[F]$. Then M' and M'' are equivalent iff $H \Rightarrow (w' \equiv w'')$.*

Hence, to find if M', M'' are equivalent it suffices to take Eq out of $\exists X[Eq \wedge F]$. (Checking if M' or M'' is a constant reduces to a few simple SAT checks.) Like *CP*, the method of [2] employs cut propagation. So, we will refer to it as CP^{pqe} . By computing relations Rel_i , CP^{pqe} takes Eq out of $\exists Z[Eq \wedge F]$ incrementally, cut by cut. As before, Rel_0 is set to Eq whereas the remaining relations Rel_i are computed by a PQE solver (see Appendix D). The difference between CP^{pqe} and *CP* is threefold. First, there are *no pre-defined relationships* to find. Relations Rel_i just need to satisfy a very simple property: $\exists Z[Rel_{i-1} \wedge Rel_i \wedge F] \equiv \exists Z[Rel_i \wedge F]$. That is the next relation Rel_i makes the previous relation Rel_{i-1} redundant. Second, CP^{pqe} is *complete*. Third, as formally proved in [2], relations Rel_i become quite simple if M' and M'' are structurally similar, which makes CP^{pqe} *very efficient*.

CP^{pqe} provides a *dramatically more powerful* version of structure-aware computing than *CP* due to PQE. Recall that $F = F'(X', V', w') \wedge F''(X'', V'', w'')$. So, $Eq(V', V'') \wedge F$ is *logically equivalent* to a simpler formula F^* defined as $F'(X', V, w') \wedge F''(X'', V, w'')$. In fact, in all implementations of *CP*, F^* is used rather than $Eq \wedge F$. However, the method of [2] *cannot* be formulated in terms of F^* since it exploits the structural peculiarity of the formula $Eq \wedge F$.

In [2], we ran some experiments with CP^{pqe} where circuits M', M'' containing a multiplier of various sizes were checked for equivalence. (The size of the multiplier ranged from 10 to 16 bits. We used an optimized version of *DS-PQE* as a PQE solver.) M', M'' were intentionally designed so that they were structurally similar but did not have any functionally equivalent points. A high-quality tool called ABC [23] showed very poor performance, whereas CP^{pqe} solved all ex-

amples efficiently. In particular, CP^{pqc} solved the example involving a 16-bit multiplier in 70 seconds, whereas ABC failed to finish it in 6 hours.

6 Model Checking And Structure-Aware Computing

In the next two sections we show that PQE can be used to design *new* methods of structure-aware computing. In this section, we apply structure-aware computing by PQE to finding the reachability diameter, i.e., to a problem of model checking.

6.1 Motivation and some background

An obvious application for an efficient algorithm for finding the reachability diameter is as follows. Suppose one knows that the reachability diameter of a sequential circuit N is less or equal to k . Then, to verify *any* invariant of N , it suffices to check if it holds for the states of N reachable in at most k transitions. This check can be done by bounded model checking [24]. Finding the reachability diameter of a sequential circuit by existing methods essentially requires computing the set of all reachable states [25,26], which does not scale well. An upper bound on the reachability diameter called the recurrence diameter can be found by a SAT-solver [27]. However, this upper bound is very imprecise. Besides, its computing does not scale well either.

6.2 Some definitions

Let $T(S', S'')$ denote the transition relation of a sequential circuit N where S', S'' are the sets of present and next state variables. Let formula $I(S)$ specify the initial states of N . (A **state** is a full assignment to the set of state variables.) A state \vec{s}_k of N with initial states I is called **reachable** in k transitions if there is a sequence of states $\vec{s}_0, \dots, \vec{s}_k$ such that $I(\vec{s}_0) = 1$ and $T(\vec{s}_{i-1}, \vec{s}_i) = 1$, $i = 1, \dots, k$. For the reason described in the remark below, we assume that N can **stutter**. That is, $T(\vec{s}, \vec{s}) = 1$ for every state \vec{s} . (If N lacks stuttering, it can be easily introduced.)

Remark 1. If N can stutter, the set of states of N reachable in k transitions is the same as the set of states reachable in *at most* k transitions. This nice property holds because, due to the ability of N to stutter, each state reachable in p transitions is also reachable in k transitions where $k > p$.

Let R_k be a formula specifying the set of states of N reachable in k transitions. That is $R_k(\vec{s}) = 1$ iff \vec{s} is reachable in k transitions. Formula $R_k(S_k)$ can be computed by performing QE on $\exists S_{0,k-1}[I_0 \wedge T_{0,k-1}]$ where $S_{0,k-1} = S_0 \cup \dots \cup S_{k-1}$ and $T_{0,k-1} = T(S_0, S_1) \wedge \dots \wedge T(S_{k-1}, S_k)$. We will call $Diam(I, N)$ the **reachability diameter** of N with initial states I if any reachable state of N requires at most $Diam(I, N)$ transitions to reach it.

6.3 Computing reachability diameter

In this subsection, we consider the problem of deciding if $\text{Diam}(I, N) \leq k$. A straightforward way to solve this problem is to compute R_k and R_{k+1} by performing QE as described above. $\text{Diam}(I, N) \leq k$ iff R_k and R_{k+1} are equivalent. Unfortunately, computing R_k, R_{k+1} even for a relatively small value of k can be very hard or simply infeasible for large circuits. Below, we show that one can, arguably, solve this problem more efficiently using *structure-aware computing*.

Proposition 4. *Let $k \geq 0$. Let $\exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$ be a formula where I_0 and I_1 specify the initial states of N in terms of variables of S_0 and S_1 respectively, $S_{0,k} = S_0 \cup \dots \cup S_k$ and $T_{0,k} = T(S_0, S_1) \wedge \dots \wedge T(S_k, S_{k+1})$. Then $\text{Diam}(I, N) \leq k$ iff I_1 is redundant in $\exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$.*

Proposition 4 reduces checking if $\text{Diam}(I, N) \leq k$ to finding if I_1 is redundant in $\exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$ (which is the *decision version* of PQE). Note that the presence of I_1 simply “cuts out” the initial time frame (indexed by 0). So, *semantically*, $\exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$ is equivalent to $\exists S_{1,k}[I_1 \wedge T_{1,k}]$ i.e., specifies the states reachable in k transitions. But the former has a different *structure* one can exploit by PQE. Namely, proving I_1 redundant in $\exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$ means that the states reachable in k and $k + 1$ transitions are the same. Importantly, I_1 is a *small* piece of the formula. So, proving it redundant can be much more efficient than computing R_k and R_{k+1} . For instance, computing R_{k+1} by QE requires proving the *entire formula* $I_0 \wedge T_{0,k}$ redundant in $R_{k+1} \wedge \exists S_{0,k}[I_0 \wedge T_{0,k}]$.

7 SAT And Structure-Aware Computing

In this section, we relate structure-aware computing by PQE to the satisfiability problem (SAT). Given a formula $F(X)$, SAT is to check if F is satisfiable i.e., whether $\exists X[F]=1$.

7.1 Motivation, some background, formulas with structure

SAT plays a huge role in practical applications. Modern SAT solvers are descendants of the DPLL procedure [28] that checks the satisfiability of a formula $F(X)$ by looking for a satisfying assignment. They identify subspaces where F is unsatisfiable due to an assignment conflict and learn conflict clauses to avoid those subspaces (see e.g., [16,29,30,31]). If a satisfying assignment is found, then $\exists X[F] = 1$. Otherwise, $\exists X[F] = 0$.

Our motivation for applying PQE to solving SAT is as follows. The descendants of DPLL implicitly perform QE on $\exists X[F]$ and, as we mentioned earlier, QE is a *semantic* operation. That is, if $\exists X[F] = b$ where $b \in \{0, 1\}$ and $F' \equiv F$, then $\exists X[F'] = b$. Below, we show that one can solve SAT by checking redundancy of a *subset* of clauses G in $\exists X[F]$. Proving redundancy of G is a *structural* operation. (If G is redundant in $\exists X[F]$ and $F' \equiv F$, this does not entail redundancy of G in $\exists X[F']$.) So, SAT solving by PQE is formula-specific and hence facilitates *structure-aware computing*.

One of the most common types of formula-specific properties are *local* redundancies stemming from design *unobservabilities*. Suppose, for instance, that F specifies a circuit N and a variable $x \in X$ describes an input of a 2-input AND gate g of N . Consider the subspace $x = 0$. Some gates feeding the *other* input of g can become unobservable in this subspace i.e., they do not affect the output of N if $x = 0$. Then the clauses of F specifying those gates become redundant in this subspace (see Appendix E). As we argue below, a PQE based SAT-solver can easily identify and exploit local redundancies (and so design unobservabilities).

7.2 Reducing SAT to the decision versions of QE and PQE

Proposition 5 below reduces SAT to the “decision version” of QE formulated in terms of redundancy based reasoning.

Proposition 5. *Formula $F(X)$ is satisfiable iff F is redundant in $\exists X[F]$.*

Now we reduce SAT to the decision version of PQE.

Proposition 6. *Let $F(X)$ be a formula and \vec{x} be a full assignment to X . Let G denote the set of clauses of F falsified by \vec{x} . Formula F is satisfiable iff G is redundant in $\exists X[F]$.*

As we mentioned above, Proposition 6 facilitates structure-aware SAT algorithms. Besides, the PQE problem of Proposition 6 is easier than QE of Proposition 5 in the following sense. In Proposition 5, one has to check if *all* clauses are redundant in $\exists X[F]$. Proposition 6 requires doing this *only* for the subset G of F . Moreover, one does not need to prove redundancy of G *globally*. It suffices to show that G is redundant in $\exists X[F]$ in some *subspace* \vec{q} where $\vec{q} \subseteq \vec{x}$. Then F is satisfiable in subspace \vec{q} .

7.3 Solving SAT by PQE and structure-aware computing

Let SAT^{pqe} be an algorithm solving SAT by applying Proposition 6 i.e., by PQE⁴. In this subsection, we briefly discuss why using an efficient implementation of SAT^{pqe} could be beneficial for solving SAT problems possessing some structure (see Subsection 7.1). For the sake of clarity, we assume that SAT^{pqe} is similar to *DS-PQE* sketched in Section 3. Due to the specifics of the SAT problem, SAT^{pqe} terminates when it proves redundancy of G in $\exists X[F]$ in a subspace $\vec{q} \subseteq \vec{x}$ or when it adds to G an empty clause. (By definition of the set G , every new conflict clause falsified by \vec{x} has to be added to it.) In the latter case, F is unsatisfiable.

As we saw in Section 3, the set of clauses G is temporarily extended when a clause of G becomes unit. So, in general, in addition to proving redundancy of

⁴ For the sake of simplicity, we use here a rather naive SAT^{pqe} where the assignment \vec{x} never changes. In reality, one can have a much more sophisticated algorithm. For instance, one can dynamically change \vec{x} and hence the set G to take into account learned conflict clauses and/or value assignments made.

the set G (that can be very small) SAT^{pqe} may need to prove *local* redundancies for clauses of $F \setminus G$. So, derivation of design *unobservabilities* expressed via local redundancy of clauses in $\exists X[F]$ is a *natural* part of SAT^{pqe} . Those redundancies are expressed in the form of D-sequents that can be reused [32] to boost the efficiency of PQE (like conflict clauses are reused by SAT solvers). Such reusing allows SAT^{pqe} to ignore the clauses proved redundant in the current subspace earlier i.e., exploit design unobservabilities.

8 Conclusions

Virtually all efficient algorithms of hardware verification use some form of structure-aware computing (SAC). We relate SAC to Partial Quantifier Elimination (PQE). The latter is a generalization of quantifier elimination where a *part* of the formula can be taken out of the scope of quantifiers. We show that interpolation, an instance of SAC, is as a special case of PQE. We apply SAC by PQE to property generation, equivalence checking, model checking, and SAT solving.

SAC by PQE allows to introduce a generalization of testing (simulation) called property generation where one identifies a bug by producing an *unwanted design property*. SAC by PQE facilitates constructing an equivalence checker that exploits the similarity of the circuits to compare *without* searching for some predefined relations between internal points of those circuits. In model checking, SAC by PQE enables a procedure that finds the reachability diameter *without* computing the set of all reachable states. Finally, SAC by PQE facilitates building SAT-solvers that can exploit *design unobservabilities* pervasive in real-life formulas. The results above suggest that studying PQE and designing fast PQE solvers is of *great importance*.

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Appendix

A Some Examples Of Structure-Aware Computing

As mentioned in the introduction, virtually all successful techniques of hardware verification use some form of structure-aware computing. In this appendix, we put together a few examples of that. In subsections concerning testing and equivalence checking we just repeat the relevant pieces of Subsections 5.2 and 5.3.

A.1 Testing

Testing is a ubiquitous technique of hardware verification [33,34,35]. One of the reasons for such omnipresence is that testing is surprisingly effective taking into account that only a minuscule part of the truth table is sampled. This effectiveness can be explained by the fact that current testing procedures check a *particular implementation* rather than sample the truth table. This is achieved by using some coverage metric that directs test generation at invoking a certain set of events. (For instance, one may try to generate a set of tests detecting a certain set of artificial faults introduced into the circuit at hand.) So, testing can be viewed as an instance of *structure-aware computing*.

A.2 Equivalence checking

Equivalence checking is one of the most efficient techniques of formal hardware verification [36]. Let N' and N'' be the circuits to check for equivalence. In general, equivalence checking is a hard problem that does not scale well even for combinational circuits. Fortunately, in practice, N' and N'' are *structurally similar*. In this case, one can often prove the equivalence of N' and N'' quite efficiently by using structure-aware computing. The latter is to locate internal points of N' and N'' linked by simple relations like equivalence [37,38,20,21,22].

The computation of these relations moves from inputs to outputs until the equivalence of the corresponding output variables of N' and N'' is proved. (If N' and N'' are sequential circuits, relations between internal points are propagated over multiple time frames.)

A.3 Model checking

A significant boost in hardware model checking has been achieved due to the appearance of IC3 [39]. The idea of IC3 is as follows. Let N be a sequential circuit. Let $P(S)$ be an invariant to prove where S is the set of state variables of N . (Proving P means showing that it holds in every reachable state of N .) IC3 looks for an *inductive* invariant P' such that $I \Rightarrow P' \Rightarrow P$ where $I(S)$ specifies the initial states of N . IC3 builds P' by constraining P via adding so-called inductive clauses. The high scalability of IC3 can be attributed to the fact that in many cases P is “almost” *inductive*. So, to turn P into P' , it suffices to add a relatively small number of clauses. Building P' as a variation of P can be viewed as a form of *structure-aware computing*.

A.4 SAT solving

The success of modern SAT-solvers can be attributed to two techniques. The first technique is the conflict analysis introduced by GRASP [16]. The idea is that when a conflict occurs in the current subspace, one identifies the set of clauses responsible for this conflict. This set is used to generate a so-called conflict clause that is falsified in the current subspace. So, adding it to the formula diverts the SAT solver from any subspace where the same conflict occurs. Finding clauses involved in a conflict can be viewed as a form of *structure-aware computing*.

The second key technique of SAT solving introduced by Chaff [29] is to employ decision making that involves variables of recent conflict clauses. The reason why Chaff-like decision making works so well can be explained as follows. Assume that a SAT-solver with conflict clause learning checks the satisfiability of a formula F . Assume that F is unsatisfiable. (If F is satisfiable, the reasoning below can be applied to every subspace visited by this SAT-solver where F was unsatisfiable.) Learning and adding conflict clauses produces an unsatisfiable core of F that includes learned clauses. This core gradually shrinks in size and eventually reduces to an empty clause. The decision making of Chaff helps to identify the subset of clauses/variables of F making up the ever-changing unsatisfiable core that incorporates conflict clauses. So, one can also view the second technique as a form of *structure-aware computing*.

B Proofs Of Propositions

Proposition 1. *Let a clause C be blocked in a formula $F(X, Y)$ with respect to a variable $x \in X$. Then C is redundant in $\exists X[F]$ i.e., $\exists X[F \setminus \{C\}] \equiv \exists X[F]$.*

Proof. It was shown in [4] that adding a clause $B(X)$ blocked in $G(X)$ to the formula $\exists X[G]$ does not change the value of this formula. This entails that removing a clause $B(X)$ blocked in $G(X)$ does not change the value of $\exists X[G]$ either. So, B is redundant in $\exists X[G]$.

Let \vec{y} be a full assignment to Y . Then the clause C of the proposition at hand is either satisfied by \vec{y} or $C_{\vec{y}}$ is blocked in $F_{\vec{y}}$ with respect to x . (The latter follows from the definition of a blocked clause.) In either case, $C_{\vec{y}}$ is redundant in $\exists X[F_{\vec{y}}]$. Since this redundancy holds in every subspace \vec{y} , the clause C is redundant in $\exists X[F]$.

Proposition 2. *Formula $H(Y)$ is a solution to the PQE problem of taking a clause C out of $\exists X[F(X, Y)]$ (i.e., $\exists X[F] \equiv H \wedge \exists X[F \setminus \{C\}]$) iff*

1. $F \Rightarrow H$ and
2. $H \wedge \exists X[F] \equiv H \wedge \exists X[F \setminus \{C\}]$

Proof. The if part. Assume that conditions 1, 2 hold. Let us show that $\exists X[F] \equiv H \wedge \exists X[F \setminus \{C\}]$. Assume the contrary i.e., there is a full assignment \vec{y} to Y such that $\exists X[F] \neq H \wedge \exists X[F \setminus \{C\}]$ in subspace \vec{y} .

There are two cases to consider here. First, assume that F is satisfiable and $H \wedge \exists X[F \setminus \{C\}]$ is unsatisfiable in subspace \vec{y} . Then there is an assignment (\vec{x}, \vec{y}) satisfying F (and hence satisfying $F \setminus \{C\}$). This means that (\vec{x}, \vec{y}) falsifies H and hence F does not imply H . So, we have a contradiction. Second, assume that F is unsatisfiable and $H \wedge \exists X[F \setminus \{C\}]$ is satisfiable in subspace \vec{y} . Then $H \wedge \exists X[F]$ is unsatisfiable too. So, condition 2 does not hold and we have a contradiction.

The only if part. Assume that $\exists X[F] \equiv H \wedge \exists X[F \setminus \{C\}]$. Let us show that conditions 1 and 2 hold. Assume that condition 1 fails i.e., $F \not\Rightarrow H$. Then there is an assignment (\vec{x}, \vec{y}) satisfying F and falsifying H . This means that $\exists X[F] \neq H \wedge \exists X[F \setminus \{C\}]$ in subspace \vec{y} and we have a contradiction. To prove that condition 2 holds, one can simply multiply both sides of the equality $\exists X[F] \equiv H \wedge \exists X[F \setminus \{C\}]$ by H .

Proposition 3. *Assume M', M'' do not implement a constant (0 or 1). Assume $\exists Z[Eq \wedge F] \equiv H \wedge \exists Z[F]$. Then M' and M'' are equivalent iff $H \Rightarrow (w' \equiv w'')$.*

Proof. The if part. Assume that $H \Rightarrow (w' \equiv w'')$. From Proposition 2 it follows that $(Eq \wedge F) \Rightarrow H$. So $(Eq \wedge F) \Rightarrow (w' \equiv w'')$. Recall that $F = F' \wedge F''$ where F' and F'' specify M' and M'' respectively. So, for every pair of inputs \vec{v}' and \vec{v}'' satisfying $Eq(V', V'')$ (i.e., $\vec{v}' = \vec{v}''$), M' and M'' produce identical values of w' and w'' . Hence, M' and M'' are equivalent.

The only if part. Assume the contrary i.e., M' and M'' are equivalent but $H(w', w'') \not\Rightarrow (w' \equiv w'')$. There are two possibilities here: $H(0, 1) = 1$ or $H(1, 0) = 1$. Consider, for instance, the first possibility i.e., $w' = 0, w'' = 1$. Since, M' and M'' are not constants, there is an input \vec{v}' for which M' outputs 0 and an input \vec{v}'' for which M'' outputs 1. This means that the formula $H \wedge \exists Z[F]$ is satisfiable in the subspace $w' = 0, w'' = 1$. Then the formula

$\exists Z[Eq \wedge F]$ is satisfiable in this subspace too. This means that there is an input \vec{v} under which M' and M'' produce $w' = 0$ and $w'' = 1$. So, M' and M'' are inequivalent and we have a contradiction.

Proposition 4. *Let $k \geq 0$. Let $\exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$ be a formula where I_0 and I_1 specify the initial states of N in terms of variables of S_0 and S_1 respectively, $S_{0,k} = S_0 \cup \dots \cup S_k$ and $T_{0,k} = T(S_0, S_1) \wedge \dots \wedge T(S_k, S_{k+1})$. Then $\text{Diam}(I, N) \leq k$ iff I_1 is redundant in $\exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$.*

Proof. The if part. Assume that I_1 is redundant in $\exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$ i.e., $\exists S_{0,k}[I_0 \wedge T_{0,k}] \equiv \exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$. The formula $\exists S_{0,k}[I_0 \wedge T_{0,k}]$ is logically equivalent to R_{k+1} specifying the set of states of N reachable in $k+1$ transitions. On the other hand, $\exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$ is logically equivalent to R_k (because I_0 and $T(S_0, S_1)$ are redundant in $\exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$). So, redundancy of I_1 means that R_k and R_{k+1} are logically equivalent and hence, $\text{Diam}(I, N) \leq k$.

The only if part. Assume the contrary i.e., $\text{Diam}(I, N) \leq k$ but I_1 is not redundant in $\exists S_{0,k}[I_0 \wedge I_1 \wedge T_{0,k}]$. Then there is an assignment $\vec{p} = (\vec{s}_0, \dots, \vec{s}_{k+1})$ such that a) \vec{p} falsifies at least one clause of I_1 ; b) \vec{p} satisfies $I_0 \wedge T_{0,k}$; c) formula $I_0 \wedge I_1 \wedge T_{0,k}$ is unsatisfiable in subspace \vec{s}_{k+1} . (So, I_1 is not redundant because removing it from $I_0 \wedge I_1 \wedge T_{m+1}$ makes the latter satisfiable in subspace \vec{s}_{k+1} .) Condition c) means that \vec{s}_{k+1} is unreachable in k transitions whereas condition b) implies that \vec{s}_{k+1} is reachable in $k+1$ transitions. Hence $\text{Diam}(I, N) > k$ and we have a contradiction.

Proposition 5. *Formula $F(X)$ is satisfiable iff F is redundant in $\exists X[F]$.*

Proof. The if part. Let F be redundant in $\exists X[F]$. Then $\exists X[F] \equiv \exists X[F \setminus F]$. Since an empty set of clauses is satisfiable, F is satisfiable too.

The only if part. Assume the contrary i.e., F is satisfiable and F is not redundant in $\exists X[F]$. This means that there is a formula H where $H \neq 1$ such that $\exists X[F] \equiv H \wedge \exists X[F \setminus F]$. Since all variables of F are quantified in $\exists X[F]$, then H is a constant. The only option here is $H = 0$. So, $\exists X[F] = 0$ and we have a contradiction.

Proposition 6. *Let $F(X)$ be a formula and \vec{x} be a full assignment to X . Let G denote the set of clauses of F falsified by \vec{x} . Formula F is satisfiable (i.e., $\exists X[F]=1$) iff the formula G is redundant in $\exists X[F]$.*

Proof. The if part. Let G be redundant in $\exists X[F]$. Then $\exists X[F] \equiv \exists X[F \setminus G]$. Since \vec{x} satisfies $F \setminus G$, then $\exists X[F \setminus G] = 1$. Hence $\exists X[F] = 1$ too.

The only if part. Assume the contrary i.e., F is satisfiable and G is not redundant in $\exists X[F]$. This means that there is a formula H where $H \neq 1$ such that $\exists X[F] \equiv H \wedge \exists X[F \setminus G]$. Since all variables of F are quantified in $\exists X[F]$, then H is a constant. The only option here is $H = 0$. So, $\exists X[F] = 0$ and we have a contradiction.

C A Single-Test Property

Let $M(X, V, W)$ be a combinational circuit where X, V, W denote the internal, input and output variables of M . Let \vec{v} be a test and \vec{w} be the output of M under \vec{v} . (Here \vec{v} is a full assignment to the *input* variables of V and \vec{w} is a full assignment to the *output* variables of W .) Let $H^{\vec{v}}(\mathbf{V}, \mathbf{W})$ be a formula such that $H^{\vec{v}}(\vec{v}', \vec{w}') = 1$ iff $\vec{v}' \neq \vec{v}$ or $(\vec{v}' = \vec{v}) \wedge (\vec{w}' = \vec{w})$. One can view $H^{\vec{v}}$ as describing the input/output behavior of M under the test \vec{v} . Let $F(X, V, W)$ be a formula specifying the circuit M as explained in Subsection 5.1. The formula $H^{\vec{v}}$ is implied by F and so it is a *property* of M . In Subsection 5.2, we refer to $H^{\vec{v}}$ as a **single-test property**.

One can obtain $H^{\vec{v}}$ by combining PQE with clause splitting. Let $V = \{v_0, \dots, v_k\}$ and C be a clause of F . Let F' denote the formula obtained from F by replacing C with the following $k+2$ clauses: $C_0 = C \vee \overline{l(v_0)}, \dots, C_k = C \vee \overline{l(v_k)}$, $C_{k+1} = C \vee l(v_0) \vee \dots \vee l(v_k)$, where $l(v_i)$ is a literal of v_i . (So, $F' \equiv F$.) Consider the problem of taking the clause C_{k+1} out of $\exists X[F']$. One can show [3] that a) this PQE problem has *linear* complexity; b) taking out C_{k+1} produces a *single-test property* $H^{\vec{v}}$ corresponding to the test \vec{v} falsifying the literals $l(v_0), \dots, l(v_k)$.

D Computing Rel_i By CP^{pqe}

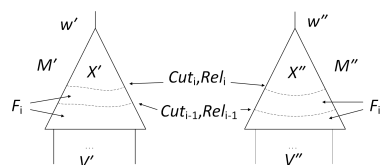


Fig. 2: Computing Rel_i in CP^{pqe}

In Subsection 5.3, we recalled CP^{pqe} , a method of equivalence checking by PQE introduced in [2]. In this appendix, we describe how CP^{pqe} computes the formula Rel_i specifying relations between cut points of Cut_i . We reuse the notation of Subsection 5.3. Let formula F_i specify the gates of M' and M'' located between their inputs and Cut_i (see Figure 2). Let Z_i denote the variables of F_i minus those of Cut_i . Then Rel_i is obtained by taking Rel_{i-1} out of $\exists Z_i[Rel_{i-1} \wedge F_i]$ i.e., $\exists Z_i[Rel_{i-1} \wedge F_i] \equiv Rel_i \wedge \exists Z_i[F_i]$. The formula Rel_i depends only on variables of Cut_i . (All the other variables of $Rel_{i-1} \wedge F_i$ are in Z_i and hence, quantified.)

Note that since Rel_i is obtained by taking out Rel_{i-1} , the latter is redundant in $\exists Z_i[Rel_{i-1} \wedge Rel_i \wedge F_i]$. One can show that this implies the property mentioned in Subsection 5.3 that Rel_{i-1} is also redundant in $\exists Z[Rel_{i-1} \wedge Rel_i \wedge F]$.

E Design unobservability And D-sequents

In this appendix, we illustrate the relation between design unobservability and D-sequents by an example. Consider the circuit $M(X, V, W)$ a fragment of which is shown in Figure 3. Here X, V, W are sets of internal, input and output variables respectively. Let $F(X, V, W) = F_{g_0} \wedge F_{g_1} \wedge F_{g_2} \wedge F_{g_3} \wedge \dots$ be a formula specifying

M where F_{g_i} describes the functionality of gate g_i . Namely, $F_{g_0} = C_0 \wedge C_1 \wedge C_2$ where $C_0 = \overline{x_0} \vee \overline{x_1} \vee x_3$, $C_1 = x_0 \vee \overline{x_3}$, $C_2 = x_1 \vee \overline{x_3}$, $F_{g_1} = C_3 \wedge C_4 \wedge C_5$ where $C_3 = \overline{x_0} \vee \overline{x_2} \vee x_4$, $C_4 = x_0 \vee \overline{x_4}$, $C_5 = x_2 \vee \overline{x_4}$, $F_{g_2} = C_6 \wedge C_7 \wedge C_8$ where $C_6 = x_3 \vee x_4 \vee \overline{x_5}$, $C_7 = \overline{x_3} \vee x_5$, $C_8 = \overline{x_4} \vee x_5$, $F_{g_3} = C_9 \wedge C_{10} \wedge C_{11}$ where $C_9 = \overline{x_5} \vee \overline{x_6} \vee x_7$, $C_{10} = x_5 \vee \overline{x_7}$, $C_{11} = x_6 \vee \overline{x_7}$.

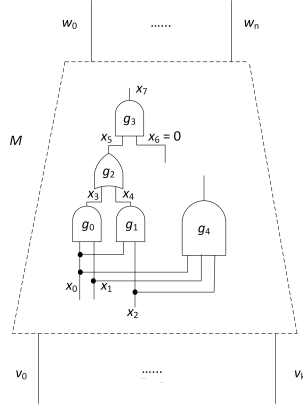


Fig. 3: Unobservable gates

Consider the problem of taking a set of clauses G out of $\exists X[F]$ where $x_i \in X, i = 0, \dots, 7$. Assume, for the sake of simplicity, that C_0, \dots, C_{11} are in G . Assume that this PQE problem is solved by *DS-PQE* and the latter is currently in subspace $\vec{r} = (x_6 = 0)$. Note that the gates g_0, g_1, g_2 are “unobservable” in this subspace. That is their values do not affect the output of M no matter what input \vec{v} is applied (as long as \vec{v} produces the assignment $x_6 = 0$). Here \vec{v} is a full assignment to $V = \{v_0, \dots, v_k\}$. Let us show that after entering the subspace \vec{r} , *DS-PQE* derives atomic D-sequents $\vec{r} \rightarrow C_i, i = 0, \dots, 8$. These D-sequents express the unobservability of g_0, g_1, g_2 in subspace \vec{r} .

First consider the clauses of F_{g_3} . Since C_9 is satisfied by \vec{r} and C_{10} is implied by the clause C_{11} in subspace \vec{r} , C_9 and C_{10} are removed $\exists X[F]$ as redundant in subspace \vec{r} . Now consider the clauses C_6, C_7, C_8 of F_{g_2} . Since the clauses C_9 and C_{10} are removed, the clauses of F_{g_2} are blocked at the variable x_5 in subspace \vec{r} . So, the D-sequents $\vec{r} \rightarrow C_i, i = 6, 7, 8$ are derived. Since the clauses of F_{g_2} are removed from the formula in subspace \vec{r} , the clauses of F_{g_0} and F_{g_1} are blocked at x_3 and x_4 respectively. So, the D-sequents $\vec{r} \rightarrow C_i, i = 0, \dots, 5$ are derived.